NON-PARABOLIC SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES

by

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CONTENTS

CHAPTER I.

SYSTEMS OF PARTIAL DIFFERENTIAL EXPRESSIONS IN FUNCTIONS OF TWO VARIABLES

1. Introduction and summary. Interest in systems of first order, linear partial differential equations for functions of two variables has been concentrated in the past on systems of totally elliptic or of totally hyperbolic character. The greatest amount of attention has gone, of course, to the Cauchy-Riemann equations in the plane or on a manifold. Beyond these, Hilbert * has discussed a problem for systems of equations of the form

$$u_x - v_y = au + bv$$

$$u_y + v_x = cu + dv$$

($u_x = \frac{\partial u}{\partial x}$, etc.), a,b,c,d being sufficiently smooth functions of x,y, in which boundary values of u are prescribed; his method required solution of certain Dirichlet and Neumann problems for the Laplace equation. Little other work is known to me on boundary problems for elliptic systems of equations in two independent variables. **

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* See [1] and also W. A. Hurwitz [1] in which Hilbert's approach is developed in greater detail.

** Petrovskii [1] (pp. 22-23) refers to work of Z. Ya. Shapiro and of N. I. Simonov in which Fredholm equations are derived for boundary problems for elliptic systems of equations with constant coefficients. It was not possible, however, to solve these Fredholm equations for bounded domains.

Hyperbolic systems recently have been extensively discussed particularly with reference to the Cauchy problems * . Systems of equations of mixed type appear to have drawn almost no attention except in a paper of T. Carleman [1] on the uniqueness of the Cauchy problem.

In the present paper, aspects of general elliptic systems of linear equations are discussed, and a beginning is also attempted of the development of a general theory of systems of mixed type. The attack stems from a well known theorem about matrices by use of which any system of first order, linear, partial differential equations in two variables can be decomposed, after linear transformations of the dependent variables and linear recombinations of the equations, into subsystems of a small number of sharply distinguished "canonical" types ** .

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* See K. O. Friedrichs [1] , Courant and Lax [1] , and the bibliography presented in Friedrichs' article.

** W. A. Hurwitz and T. Carleman also have used such decompositions in the papers cited.

The methods of the present paper are based on this canonical decomposition. They are greatly aided by the fact that any solution of a canonical system of equations can be expressed as a hypercomplex number, an element of a commutative, associative algebra. * Chapter I is devoted primarily to these matters ** .

The second chapter is concerned chiefly with canonical elliptic systems of equations which may be assumed, after a change of independent variables, to be of the form

$$u^p_x - v^p_y + au^{p+1}_x + bu^{p+1}_y - cv^{p+1}_x - dv^{p+1}_y = f^p \qquad (p = 1, \ldots, r-1)$$

$$u^p_y + v^p_x + cu^{p+1}_x + du^{p+1}_y + av^{p+1}_x + bv^{p+1}_y = g^p \qquad (\quad " \quad)$$

$$u^r_x - v^r_y = f^r$$

$$u^r_y + v^r_x = g^r,$$

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* Hypercomplex numbers have previously been used in the theory of partial differential equations for various purposes. For their application to important kinds of systems of equations with several independent variables, see the bibliography in H. G. Häfeli's paper [1] . Their use in generalizing the concepts of derivative and analyticity is discussed, and an extensive bibliography on this subject is presented, by J. A. Ward [1] . J. B. Diaz [1] has employed hypercomplex numbers to study partial differential equations in one function of two variables when the characteristic determinant is a power of a positive definite quadratic form.

** The first chapter includes also a discussion of the changes in classification effected by various methods of reduction of an equation of higher order in one unknown function to a system of equations of first order in several dependent variables.

where a,b,c,d,f,g are sufficiently regualr functions of x,y. Integral representations, analogous to Cauchy's formula, are derived for solutions of such systems of equations; the solutions of the homogeneous equations are represented in terms of arbitrary analytic functions. * The chapteris concluded with an extension of a well-known theorem of T. Carleman [1] on the uniqueness of solutions of Cauchy problems for systems of mixed hyperbolic and elliptic type.

Chapter III begins with a discussion of boundary problems for elliptic systems of equations. A minimum-maximum priciple is stated for solutions of elliptic systems of the form

$$u_x + au_y - bv_y = 0$$

$$v_x + av_y + bu_y = 0,$$

a and b being continuous functions of x,y; from this follows a uniqueness theorem for certain boundary problems for canonical elliptic systems of equations. The existence of solutions of suitable boundary problems for canonical elliptic systems of equations then is discussed with the aid of the ideas developed by E. E. Levi [1, 2] and G. Giraud [3] in their studies of elliptic equations of second order in one unknown function. The paper ends with a theorem on the existence of a solution of a mixed boundary- and initial-value problem for systems of mixed hyperbolic and elliptic type.

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* In analogy with the well-known representation of a biharmonic function as xu + v, u and v being arbitrary harmonic functions.

2. <u>Classification</u> <u>of</u> <u>differential</u> <u>forms</u>. Let

$$L\left(\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y}\right)u \equiv \left(\sum_{\lambda} a_{\lambda}\frac{\partial^{m}}{\partial x^{m-\lambda}\partial y^{\lambda}} + \sum_{s+t<m} b_{st}\frac{\partial^{s+t}}{\partial x^{s}\partial y^{t}}\right)u$$

be a linear differential form. Its principal part, defined as the sum of the terms of highest order is $P\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)u = \sum_{\lambda} a_{\lambda}\frac{\partial^{m}u}{\partial x^{m-\lambda}\partial y^{\lambda}}$. $P(\lambda,\mu)$ is called the characteristic polynomial of the form. Such a form is classified according to the factors of its characteristic polynomial which are irreducible in the real field. If these factors are all quadratic in λ,μ, the differential form is called <u>elliptic</u>; if the factors are all linear and no two proportional, the form is <u>hyperbolic</u>; if they are linear and all multiple, it is <u>parabolic</u>; otherwise, the form is of <u>mixed</u> type. An equation $Lu = f(x,y)$ is said to be of the type of the form Lu.

A somewhat analogous classification of systems of linear first-order expressions

$$L_{i}u \equiv \sum_{j=1}^{n}\left(a_{ij}\frac{\partial}{\partial x} + b_{ij}\frac{\partial}{\partial y} + c_{ij}\right)u^{j} \qquad (i = 1,\cdots,n)$$

is based upon the <u>characteristic</u> <u>matrix</u>

$$M(\lambda,\mu) = \left(a_{ij}\lambda + b_{ij}\mu\right)$$

which, in the region of the xy-plane considered, will always be assumed not to have identically vanishing determinant. Thus, it may be further assumed, at least after a rotation in the xy-plane, that $a = (a_{ij})$ is non-singular.*

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* For if $\det M(\alpha,\beta) \neq 0$, with $\alpha^2+\beta^2 = 1$, we may set $x' = \alpha x + \beta y$, $y' = -\beta x + \alpha y$. The new characteristic matrix is $(a'_{ij}\lambda' + b'_{ij}\mu')$ with $\det(a'_{ij}) = \det(\alpha a_{ij} + \beta b_{ij}) \neq 0$.

The system $L_i u$ is then called elliptic, if the elementary divisors of $a^{-1} M(\lambda, \nu^1) = (\lambda I - b^1)$ ($b^1 = a^{-1}(b_{ij})$) are all complex-valued, hyperbolic if they are all real-valued and simple, parabolic, if all the elementary divisors are real-valued and multiple, of mixed type otherwise. The determinant of the characteristic matrix is called the characteristic determinant and is in some ways analogous to the characteristic polynomial of a linear differential form in one unknown function. Thus, a system of forms is elliptic, if, and only if, the irreducible factors of the characteristic determinant are quadratic. For a system to be hyperbolic, it is sufficient, but not necessary, that the factors be all linear and no two proportional; and for it to be parabolic, it is necessary, but not sufficient, that the factors of the characteristic determinant be linear and multiple. A system of equations $L_i u = f_i$ is said to be of the type of the system of forms $L_i u$.

It is well -known ** that the classification of a linear form is unchanged by non-singular coordinate transformations. The classification of a system of first-order forms is invariant to these and also to linear transformations of the dependent variables, and to linear recombinations of the forms composing the system, assuming the transformations in each case to be non-singular.

The significance of this scheme of classification is most clearly revealed, perhaps, in such a system of linear expressions as

(2.1) $\lambda u = u_x + bu_y,$

** See Courant-Hilbert, Vol. 2, p. 139, p. 142.

where u is an n×1 matrix: $u = \begin{pmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^n \end{pmatrix}$ of the dependent variables, and b is an n×n matrix having the Jordan form. Thus, b is the direct sum:

$$b = \begin{pmatrix} b_1 & & & & \\ & b_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & b_s \end{pmatrix}$$

of submatrics of the type

$$b_k = \begin{pmatrix} z_k(x,y) & 1 & & \\ & z_k(x,y) & \ddots & \\ & & \ddots & 1 \\ & & & z_k(x,y) \end{pmatrix}$$

where the $z_k(x,y)$, which will be assumed to be continuous in a domain R of the xy-plane, may be either real-valued or complex-valued (non-real) in R. Let r_i be the order of the matrix b_i, and let

$$U^k = \begin{pmatrix} u^{r_1 + \cdots + r_k + 1} \\ \cdot \\ \cdot \\ \cdot \\ u^{r_1 + \cdots + r_{k+1}} \end{pmatrix}.$$

The given system of expressions can be then written as

$$(2.2) \qquad L^k U^k = U^k_x + b_k U^k_y \qquad (k = 1, \ldots, s).$$

We may, in general, assume the elements of U^k to be real-valued, if z_k is real-valued, complex-valued, if z_k is complex-valued. In the latter case, with $z_k = z' + iz''$ $(z'' \neq 0)$, $u^{r_k+m} = v^m + iw^m$, and $r_{k+1} = r$, the system of forms (2.2) has the components

$$L^{km} U^k = L^m(V,W) + iM^m(V,W) = v^m_x + iw^m_x + (z' + iz'')(v^m_y + iw^m_y) + v^{m+1}_y + iw^{m+1}_y \quad (m = 1, \ldots, r-1)$$

$$L^{kr} U^k = L^r(V,W) + iM^r(V,W) = v^r_x + iw^r_x + (z' + iz'')(v^r_y + iw^r_y),$$

where $V = \{v^1, \ldots v^r\}$, $W = \{w^1, \ldots, w^r\}$.

Separating real and imaginary parts, we have, equivalently, the system

$$\begin{aligned} L^m(V,W) &= v^m_x + z'v^m_y - z''w^m_y + v^{m+1}_y \qquad (m = 1, \ldots, r-1) \\ (2.3) \qquad M^m(V,W) &= w^m_x + z'w^m_y + z''v^m_y + w^{m+1}_y \qquad (\quad '' \quad) \\ L^r(V,W) &= v^r_x + z'w^r_y - z''w^r_y \\ M^r(V,W) &= w^r_x + z'w^r_y + z''v^r_y, \end{aligned}$$

which we shall call the <u>canonical</u> <u>elliptic</u> <u>form</u>. To this form can be reduced any system of expressions $Lu = au_x + bu_y$ (a non-singular) such that the 2r-rowed matrix $a^{-1}b$ has r-fold non-real-valued characteristic roots. For letting p be the matrix such that $b = p^{-1}(a^{-1}b)\,p$ is of the Jordan normal form, and introducing new dependent variables by $v = p^{-1}u$, we see that $Lv = p^{-1}a^{-1}L(pv) - p^{-1}p_x v - p^{-1}a^{-1}bp_y v = v_x + bv_y$, which is of the form (2.1).

One of the s systems of forms of (2.1), say the j-th, for which $z_j(x,y)$ is real-valued has the components

$$(2.4)\quad L^{jm}(U^j) = u_x^m + z_j u_y^m + u_y^{m+1} \quad (m = r_1 + \dots + r_j + 1,\ r_1 + \dots + r_j + 2, \dots, r_1 + \dots + r_{j+1} - 1)$$

$$L^{jt}(U^j) = u_x^t + z_j u_y^t \qquad (t = r_1 + \dots + r_{j+1}).$$

Such a system will be called the <u>canonical parabolic form</u>, if $r_{j+1} > 1$, the <u>canonical hyperbolic form</u>, if $r_{j+1} = 1$. Any system of expressions $Lu \equiv Au_x + Bu_y$ (A non-singular) such that the r-rowed ($r > 1$) matrix $A^{-1}B$ has one r-fold real-valued characteristic root can be reduced to the canonical parabolic form by the means that were used for systems of elliptic type.

In similar fashion, any system of expressions $Lu = Au_x + Bu_y$ of mixed type can be reduced to an equivalent system of the type of $\mathcal{L}u$ in (2.1), which will be called the <u>canonical form</u> of the system. A system of equations $Lu = f(x,y,u)$ will be said to be in canonical form, if the system of expressions Lu is in canonical form.

3. <u>Reduction of equations of higher order to systems of equations of first order.</u> As will become evident, the theory of systems of first-order equations in n unknown functions is closely related to the theory of n-th order equations in one function. An exact equivalence between the two, such as exists in the theory of ordinary differential equations, is lacking, however, as can be seen from the following example: to solve the hyperbolic equation $au_{xx} + 2bu_{xy} + cu_{yy} = f$ (a,b,c,f constants satisfying $ac-b^2 < 0$) prescribing $u(0,y) = A(y)$, $u_x(0,y) = B(y)$, where A and B are required, say, to have continuous second derivatives.

This specific problem is, indeed, equivalent to a problem for a system of first-order equations, to that, namely, of solving

$$U_x = V, \; V_y - W_x = 0, \; aV_x + 2bV_y + cW_y = 0$$

prescribing $$U(0,y) = A(y), \; V(0,y) = B(y), \; W(0,y) = \frac{dA(y)}{dy} .$$

Unique solutions $u(x,y)$ of the first, and $U(x,y)$, $V(x,y)$, $W(x,y)$ of the second problem exist, and $u(x,y) = U(x,y)$. The given second-order equation by itself is, however, not equivalent to the system, for if $W(0,y)$ be prescribed, say, as $\frac{dA}{dy} + C(y)$, then U fails not only to coincide with u but even to satisfy the same differential equation. Computation shows, in fact, that $aU_{xx} + 2bU_{xy} + cU_{yy} = f - cC'(y)$.

In the foregoing example, there is a unique partial differential equation, namely, $aU_{xxx} + 2bU_{xxy} + cU_{xyy} = 0$, which U must satisfy. It can, however, be shown * that this situation is exceptional: in general, each unknown function in a system of first-order equations satisfies simultaneously more than one equation of higher order. Thus, the theory of equations in one function does not include the theory of systems of equations in several functions.

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* See Courant-Hilbert, Vol. 2, p. 12, pp. 46-47.

A specific equation in one dependent variable, say

$$(3.1)\qquad \sum_{r=0}^{m} a_r \frac{\partial^m u}{\partial x^{m-r}\partial y^r} + \sum_{0\le p+q<m} b_{pq} \frac{\partial^{p+q} u}{\partial x^p \partial y^q} = f,$$

the coefficients a_r, b_{pq}, f being assumed to be continuous functions of x, y, can always be reduced to a system of first order equations in the sense that any solution of (3.1) furnishes a solution of the system. In general, such a reduction can be accomplished in a variety of ways with correspondingly different effects upon the classification of the resulting system. We may, for instance, introduce new functions v^{ij} through the equations

$$u_x = v^{10},\ v^{10}_x = v^{20},\ \dots,\ v_x^{m-2,0} = v^{m-1,0};$$

$$v^{01}_x - v^{10}_y = 0,\ v^{11}_x - v^{20}_y = 0,\ v^{21}_x - v^{30}_y = 0,\ \dots,\ v^{m-2,1}_x - v^{m-1,0}_y = 0;$$

$$v^{02}_x - v^{11}_y = 0,\ v^{12}_x - v^{21}_y = 0,\ \dots,\ v^{m-3,2}_x - v^{m-2,1}_y = 0;$$

....

$$v^{0,m-1}_x - v^{1,m-2}_y = 0,$$

i.e., with $u = v^{00}$,

$$v^{r0}_x = v^{r+1,0} \qquad (r = 0,1,\dots,m-2)$$

$$v^{rs}_x - v^{r+1,s-1}_y = 0 \quad (r + s = 1,\dots,m-1 \text{ with } r \text{ running through } 0,\dots,m-2, \text{ and } s \text{ " " } 1,\dots,m-1).$$

After the imposition of proper initial conditions, say

$$v^{rs}(0,y) = \frac{d^s}{dy^s}\left(\frac{\partial^r u(x,y)}{\partial x^r} \Bigg|_{x=0}\right),$$

we would have $v^{rs} = \dfrac{\partial^{r+s} u}{\partial x^r \, \partial y^s}$. Hence, any solution of (3.1) furnishes a solution of the system consisting of the equations (3.2) together with

$$(3.3) \qquad \sum_{r=0}^{m-1} a_r v_x^{m-r-1,r} + a_m v_y^{0,m-1} + \sum_{0 \le p+q < m} b_{pq} v^{pq} = f.$$

For this system, the characteristic matrix, its columns arranged as labelled, is

$$v^{00} \; v^{10} \; v^{01} \; v^{20} \; v^{11} \; v^{02} \; v^{30} \; v^{21} \; v^{12} \; v^{03} \; \ldots \; v^{m-1,0} \; v^{m-2,1} \; \ldots \; v^{1,m-2} \; v^{0,m-1}$$

$$\begin{pmatrix}
\lambda &&&&&&&&&&&&&& \\
& \lambda &&&&&&&&&&&&& \\
& -\mu & \lambda &&&&&&&&&&&& \\
&&& \lambda &&&&&&&&&&& \\
&&& -\mu & \lambda &&&&&&&&&& \\
&&&& -\mu & \lambda &&&&&&&&& \\
&&&&&& \lambda &&&&&&&& \\
&&&&&& -\mu & \lambda &&&&&&& \\
&&&&&&& -\mu & \lambda &&&&&& \\
&&&&&&&& -\mu & \lambda &&&&& \\
&&&&&&&&&& \ddots &&&& \\
&&&&&&&&&&& -\mu & \lambda && \\
&&&&&&&&&&& \vdots & -\mu & \ddots & \\
&&&&&&&&&&& \vdots & \vdots & -\mu & \lambda \\
&&&&&&&&&&& a_0\lambda & a_1\lambda & \cdot \; a_{m-2}\lambda & (a_{m-1}\lambda + a_m\mu)
\end{pmatrix}$$

Hence the characteristic determinant of the system is the product of the determinant

$$\begin{vmatrix} -\mu & \lambda & & & & \\ & -\mu & \lambda & & & \\ \vdots & & & \ddots & & \\ & & & & \ddots & \lambda \\ a_0\lambda & a_1\lambda & a_2\lambda & \cdots & a_{m-2}\lambda & (a_{m-1}\lambda + a_m\mu) \end{vmatrix}$$

by a power of λ, the power being equal to the total number of functions v^{rs} $(0 \leq r + s \leq m-1)$, namely to $m(m+1)/2$. Hence, by a well-known result of determinant theory * the characteristic determinant for the system is equal to $-\lambda^{m(m+1)/2} P(\lambda, \mu)$, where $P(\lambda, \mu) = \sum_0^m a_r \lambda^{m-r} \mu^r$ is the characteristic polynomial for the original equation. We note also that the elementary divisors corresponding to the $\frac{m(m+1)}{2}$ - fold factor λ are simple, and it follows, in particular, that a hyperbolic equation can in the sense considered be reduced to a hyperbolic system. **

By other methods, elliptic equations can be reduced to elliptic systems. Any solution, for example, of the second-order equation

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g,$$

the coefficients being functions of x, y, furnishes a solution of the

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* See, for example, A. A. Albert, pp. 80-81 for a simple proof by induction.

** This method of reduction is essentially that employed in Courant-Hilbert [1] Vol. 2, pp. 145-6 in connection with second-order equations in several variables.

system

$$u^0_x - v_y = u^{10}$$

$$u^0_y + v_x = u^{01}$$

$$u^{10}_y - u^{01}_x = 0$$

$$au^{10}_x + bu^{10}_y + cu^{01}_y + du^{10} + eu^{01} + fu = g,$$

in which to see this we need merely identify u^0 with u and v with zero.

To deal with an m-th-order equation of form (3.1), let us consider the system S consisting of the sets of equations $S^0, \ldots, S^m$ defined as follows:

S^0:
$$u^{00}_x - v^{00}_y = u^{10}$$
$$u^{00}_y + v^{00}_x = u^{01}$$

S^1:
$$u^{10}_x - v^{10}_y = u^{20}, \quad u^{01}_x - v^{01}_y - u^{10}_y = 0$$
$$u^{10}_y + v^{10}_x = u^{11}, \quad u^{01}_y + v^{01}_x = u^{02}$$

S^2:
$$u^{20}_x - v^{20}_y = u^{30}, \quad u^{11}_x - v^{11}_y - u^{20}_y = 0, \quad u^{02}_x - v^{02}_y = u^{12}$$
$$u^{20}_y + v^{20}_x = u^{21}, \quad u^{11}_y + v^{11}_x - u^{02}_x = 0, \quad u^{02}_y + v^{02}_x = u^{03}$$

S^3:
$$u^{30}_x - v^{30}_y = u^{40}, \quad u^{12}_x - v^{12}_y = u^{22}, \quad u^{21}_x - v^{21}_y - u^{30}_y = 0, \quad u^{03}_x - v^{03}_y - u^{12}_y = 0,$$
$$u^{30}_y + v^{30}_x = u^{31}, \quad u^{12}_y + v^{12}_x = u^{13}, \quad u^{21}_y - v^{21}_x - u^{12}_x = 0, \quad u^{03}_y + v^{03}_x = u^{04},$$

. . . .

$$
S^i; \quad (i < m) \quad \left\{\begin{array}{l}
\left.\begin{array}{l} u_x^{i-2j,\,2j} - v_y^{i-2j,\,2j} = u^{i-2j+1,\,2j} \\ u_y^{i-2j,\,2j} + v_x^{i-2j,\,2j} = u^{i-2j,\,2j+1} \end{array}\right\} \begin{array}{l} (j = 0, 1, \ldots, (1/2)\, i \text{ for } i \text{ even}, \\ \quad 0, 1, \ldots, (1/2)(i-1) \text{ for } i \text{ odd}) \end{array} \\
\left.\begin{array}{l} u_x^{i-1-2k,\,2k+1} - v_y^{i-1-2k,\,2k+1} - u_y^{i-2k,\,2k} = 0 \\ u_y^{i-1-2k,\,2k+1} + v_y^{i-1-2k,\,2k+1} - u_x^{i-2k-2,\,2k+2} = 0 \end{array}\right\} \begin{array}{l} k = 0, 1, \ldots, (1/2)(i-2) \text{ for } i \text{ even} \\ \quad 0, 1, \ldots, (1/2)(i-3) \text{ for } i \text{ odd}) \end{array} \\
\left.\begin{array}{l} u_x^{0i} - v_y^{0i} - u_y^{1,i-1} = 0 \\ u_y^{0i} + v_x^{0i} = u^{0,i+1} \end{array}\right\} \text{ for } i \text{ odd;}
\end{array}\right.
$$

$$
S^m; \quad u_x^{m-1-j,\,j} - u_y^{m-j,\,j-1} = 0 \qquad (j = 0, 1, \ldots, m-1),
$$

$$
\sum_{r=0}^{m-1} a_r u_x^{m-r-1,r} + a_m u_y^{0,m-1} + \sum_{0 \leq p+q < m} b_{pq} u^{pq} = f.
$$

Let us observe at the outset that a solution u of (2.1) leads to a solution of this system S through setting $u^{00} = u$, $v^{ij} = 0$. All that remains is to show then that the number of real characteristic values is not greater for the characteristic determinant of the system than for the characteristic polynomial of the equation (2.1). To this effect, we note first that the characteristic matrix of the system S is the direct sum of the characteristic matrices for the sybsystems S^i, and, hence, that the characteristic determinant for S is the product of the characteristic determinants M^i of the S^i. With columns arranged as labelled, $M^i =$

$$
\begin{array}{c}
u^{i0}\quad v^{i0}\quad u^{i-1,1}\quad v^{i-1,1}\quad \cdots \quad u^{1,i-1}\quad v^{1,i-1}\quad u^{0i}\;,\; v^{0i} \\
\left|\begin{array}{cccccccc|cc}
 & & & & & & & & 0 & 0 \\
 & & & & M^{i-1} & & & & 0 & 0 \\
 & & & & & & & & -\lambda & 0 \\
\hline
0 & 0 & 0 & 0 & \vdots & 0 & 0 & & \lambda & -\mu \\
0 & 0 & 0 & 0 & \cdot & 0 & 0 & & \mu & \lambda
\end{array}\right|
\end{array}
\quad (\lambda^2+\mu^2)M^{i-1}
$$

for even $i < m$, and $M^i =$

$$
\left|\begin{array}{ccccc|cc}
 & & & & & 0 & 0 \\
 & & M^{i-1} & & & & \\
 & & & & & 0 & 0 \\
\hline
0 & \cdots & 0 & -\mu & 0 & \lambda & -\mu \\
0 & \cdots & 0 & 0 & 0 & \mu & \lambda
\end{array}\right|
\quad (\lambda^2+\mu^2)M^{i-1}
$$

for odd $i < m$. It follows by induction that $M^i = (\lambda^2 + \mu^2)^{i+1}$ for $i < m$. Since, as noted above,

$$M^m = \sum_{r=0}^{m-1} a_r \lambda^{m-r} \mu^r = P(\lambda, \mu),$$

the characteristic polynomial of the equation (2.1), it follows that the characteristic determinant of the system S is

$$(\lambda^2 + \mu^2)^{m(m+1)/2} P(\lambda, \mu).$$

As with the first method of reduction of a higher-order equation to a first-order system, the system, in general, possesses solutions which do not correspond to any solution of the original equation: the system and the equation are not equivalent. Just as a Cauchy problem for a hyperbolic equation is equivalent, however, to a suitable Cauchy problem for the corresponding hyperbolic system, so is a certain boundary problem for an elliptic equation equivalent to a suitable boundary problem for the corresponding elliptic system. Let us consider, for illustration, the second-order equation $au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$ and its corresponding system as given on p. 14. For the equation, we prescribe $u = \tilde{u}$, for the system, $u^{00} = \tilde{u}$, $v = 0$ * on the boundary. Since, by virtue of the third equation of the system, v is harmonic, we have $v \equiv 0$, and the equivalence follows. Similar considerations apply to higher-order equations, but these will not be developed here.

There is a third method of reduction of an equation to a first-order system which is probably more useful than the preceding ones when it can be employed. It applies to homogeneous equations with constant coefficients in which only highest-order derivatives of the unknown function appear, and the characteristic determinant of the system it produces is equal to the characteristic polynomial of the given equation. We give the method for an equation of even order, which may be

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* It will be recognized that this boundary problem for the system is not a usual one. Ordinarily, one function from each of the pairs u^{00}, v and u^{10}, u^{01} would be prescribed.

written as

$$(3.4)\qquad \prod_{i=1}^{n}\left(a_i \frac{\partial^2}{\partial x^2} + 2b_i \frac{\partial^2}{\partial x\, \partial y} + c_i \frac{\partial^2}{\partial y^2}\right) u = 0,$$

the a_i, b_i, c_i being constants. We shall show there exist functions u^i, v^i ($i = 1,\ldots,n$) with $u = u^1$ satisfying the system

$$
\begin{aligned}
(3.5)\qquad E_1(u^1, v^1, u^2) &\equiv a_1 u^1_x + b_1 u^1_y - v^1_y + u^2_y = 0 \\
F_1(u^1, v^1, v^2) &\equiv b_1 u^1_x + c_1 u^1_y + v^1_x + v^2_y = 0 \\
E_2(u^2, v^2, u^3) &\equiv a_2 u^2_x + b_2 u^2_y - v^2_y + u^3_y = 0 \\
F_2(u^2, v^2, v^3) &\equiv b_2 u^2_x + c_2 u^2_y + v^2_x + v^3_y = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
E_{n-1}(u^{n-1}, v^{n-1}, u^n) &\equiv a_{n-1} u^{n-1}_x + b_{n-1} u^{n-1}_y - v^{n-1}_y + u^n_y = 0 \\
F_{n-1}(u^{n-1}, v^{n-1}, v^n) &\equiv b_{n-1} u^{n-1}_x + c_{n-1} u^{n-1}_y + v^{n-1}_x + v^n_y = 0 \\
E_n(u^n, v^n) &\equiv a_n u^n_x + b_n u^n_y - v^n_y = 0 \\
F_n(u^n, v^n) &\equiv b_n u^n_x + c_n u^n_y + v^n_x = 0.
\end{aligned}
$$

First, the characteristic matrix is

$$
\begin{array}{cccccccccc}
u^1 & v^1 & u^2 & v^2 & u^3 & v^3 & u^{n-1} & v^{n-1} & u^n & v^n \\
\hline
a_1\lambda + b_1\mu & -\mu & \mu & 0 & & & & & & \\
b_1\lambda + c_1\mu & \lambda & 0 & \mu & & & & & & \\
 & & a_2\lambda + b_2\mu & -\mu & \mu & 0 & & & & \\
 & & b_2\lambda + c_2\mu & \lambda & 0 & \mu & & & & \\
 & & & & & \cdots & a_{n-1}\lambda + b_{n-1}\mu & -\mu & \mu & 0 \\
 & & & & & & b_{n-1}\lambda + c_{n-1}\mu & \lambda & 0 & \mu \\
 & & & & & & & & a_n\lambda + b_n\mu & -\mu \\
 & & & & & & & & b_n\lambda + c_n\mu & \lambda
\end{array}
$$

and the characteristic determinant, therefore

$$
\prod_{i=1}^{n} \begin{vmatrix} a_1\lambda + b_1\mu & -\mu \\ b_1\lambda + c_1\mu & \lambda \end{vmatrix} = \prod_{i=1}^{n} (a_1\lambda^2 + 2b_1\lambda\mu + c_1\mu^2).
$$

Secondly, if $(u^1, \ldots, v^n)$ is a solution of this system, then $u^1, \ldots, v^n$ individually must also satisfy the original equation for u: in particular, $A_1 A_2 \ldots A_n u_1 = 0$, and $A_1 \ldots A_n v_1 = 0$, where $A_1 = a_1 \frac{\partial^2}{\partial x^2} + 2b \frac{\partial^2}{\partial x\,\partial y} + c_1 \frac{\partial^2}{\partial y^2}$.

To prove this, we first note that from

$\frac{\partial}{\partial x} E_n + \frac{\partial}{\partial y} F_n = 0$ and $(b_n \frac{\partial}{\partial x} + c_n \frac{\partial}{\partial y}) E_n - (a_n \frac{\partial}{\partial x} + b_n \frac{\partial}{\partial y}) F_n = 0$ follows $A_n u^n = 0$ and $A_n v^n = 0$. Let us now assume for induction that $A_{i+1} \ldots A_n u^{i+1} = A_{i+1} \ldots A_n v^{i+1} = 0$ $(i < n)$.

From $E_i = 0$, $F_i = 0$ we have $A_i u^i + u^{i+1}_{xy} + v^{i+1}_{yy} = 0$, and

$$A_i v^i - (b_i \frac{\partial}{\partial x} + c_i \frac{\partial}{\partial y}) u^{i+1}_y + (a_i \frac{\partial}{\partial x} + b_i \frac{\partial}{\partial y}) v^{i+1}_y = 0,$$

and applying to each of these equations the operator $A_{i+1} \ldots A_n$ gives us, in view of the induction assumption, $A_i \ldots A_n u^i = A_i \ldots A_n v^i = 0$. Thus, in particular, $A_1 \ldots A_n u^1 = 0$.

Finally, we must show that if u^1 is any solution of the equation (3.4), there exist functions u^i, v^i satisfying the system (2.5). For this, we shall emply a rather special result on the compatibility of simultaneous partial differential equations as formulated in

LEMMA 3.1. Let $A_m = \sum_{i+j=m} a_{ij} D^i_x D^j_y$, $B_m = \sum_{i+j=m} b_{ij} D^i_x D^j_y$, $(D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y})$

where the $a_{0m}, \ldots, b_{m0}$ are constants. Let $f(x,y)$, $g(x,y)$ be of class $C^{(m)}$ in a convex domain R. Then there exists a function $u(x,y)$ of class $C^{(2m)}$ in R satisfying the simulatneous equations $A_m u = f$, $B_m u = g$, if and only if the <u>relation</u> <u>of</u> <u>compatibility</u>

$$A_m g - B_m f = 0$$

is satisfied.

<u>Proof:</u> The necessity of the compatibility condition is obvious. In proving the condition is sufficient, we may suppose not every a_{ij} is proportional to the corresponding b_{ij}. We may then, further, suppose

$a_{om}:b_{om} \neq a_{mo}:b_{mo}$;* and, hence, that $a_{om} = 0$, $b_{mo} = 0$ **.

It is, thus sufficient to prove the lemma for equations of the form

$$A_{m-1}D_x u = f, \quad B_{m-1}D_y u = g,$$

where A_{m-1}, B_{m-1} are any homogeneous polynomials in D_x, D_y with constant coefficients and of degree $m-1$. We observe first that the lemma is correct for $m = 1$. If it holds for $m-1$, a function v of class C^{2m-2} exists satisfying the equations $A_{m-1}v = f_y$, $B_{m-1}v = g_x$, if and only if $A_{m-1}D_x g - B_{m-1}D_y f = 0$. Supposing v to exist, define

$$U = \int_0^y \int_0^x v(\xi, \eta)\, d\xi\, d\eta,$$

---

* In the contrary case, a suitable affine transformation

$X = ax + by$, $Y = cx + dy$ $(ad-bc \neq 0)$ results in

$$A_m = \sum_{i+j=m} a_{ij} \quad (aD_X + cD_Y)^i (bD_X + dD_Y)^j = \sum_{i+j=m} a_{ij}a^i b^j D_X^m + \dots + \sum_{i+j=m} a_{ij}c^i d^j D_Y^m,$$

$$B_m = \sum_{i+j=m} b_{ij}a^i b^j D_X^m + \dots + \sum_{i+j=m} b_{ij}c^i d^j D_Y^m \; ;$$

two new operators in which the coefficients of D_X^m are not proportional to the coefficients of D_Y^m.

** For if the lemma is true in this case, it holds in general. Assuming $a_{om}:b_{om} \neq a_{mo}:b_{mo}$; there exist constants α, β, γ, δ $(\alpha\delta - \beta\gamma \neq 0)$ such that $\alpha A_m + \beta B_m = A'$ and $\gamma A_m + \delta B_m = B'$, where the coefficients of D_y^m in A' and that of D_x^m in B' are zero. The equations $A_m u = f$, $B_m u = g$ are equivalent to $A'u = f'$, $B'u = g'$, where $f' = \alpha f + \beta g$, $g' = \gamma f + \delta g$. Since there exist constants a, b, c, d such that $A_m = aA' + bB'$, $B_m = cA' + dB'$, $f = af' + bg'$, $g = cf' + bg'$, the condition $A_m g - B_m f = 0$ is readily seen to be equivalent with the condition $A'g' - B'f' = 0$.

assuming the origin to be an interior point of the domain R. It follows that $D_y(A_{m-1}D_xU - f) = 0$, and $D_x(B_{m-1}D_yU - g) = 0$, or $A_{m-1}D_xU - f = p(x)$, $B_{m-1}D_yU - g = q(y)$. Let $P(x)$, $Q(y)$ be any solutions of the ordinary differential equations $A_{m-1}D_xP(x) = p(x)$, $B_{m-1}D_yQ(y) = q(y)$, resp.. It follows that the function $u = U(x,y) - P(x) - Q(y)$ is a solution of the two equations $A_{m-1}D_xu = f$, $B_{m-1}D_yu = g$, as desired.

It is convenient, before applying the lemma, to introduce the symbols $L_i = a_iD_x + b_iD_y$, $M_i = b_iD_x + c_iD_y$, the equations of the system, in this notation, being

$$
\begin{array}{lll}
E_i(u^i, v^i, u^{i+1}) & \equiv L_iu^i - D_yv^i + D_yu^{i+1} & = 0 \quad (i = 1, \ldots, n-1) \\
F_i(u^i, v^i, v^{i+1}) & \equiv M_iu^i + D_xv^i + D_yv^{i+1} & = 0 \quad (\quad " \quad) \\
E_n(u^n, v^n) & \equiv L_nu^n - D_yv^n & = 0 \\
F_n(u^n, v^n) & \equiv M_nu^n + D_xv^n & = 0.
\end{array}
$$

Given u^1 as a solution of (3.4), we shall shows how v^1 and, successively, $u^2, v^2, \ldots, u^n, v^n$ may be determined. With the notation

$$
\begin{aligned}
G_1(u,v) &\equiv L_1u - D_yv \equiv p_1u + q_1v \\
H_1(u,v) &\equiv M_1u + D_xv \equiv r_1u + s_1v \\
G_2(u,v) &\equiv L_2G_1 - D_yH_1 \equiv p_2u + q_2v \\
H_2(u,v) &\equiv M_2G_1 + D_xH_1 \equiv r_2u + s_2v
\end{aligned}
$$

....

$$
\begin{aligned}
G_i(u,v) &\equiv L_iG_{i-1} - D_yH_{i-1} \equiv p_iu + q_iv \\
H_i(u,v) &\equiv M_iG_{i-1} + D_xH_{i-1} \equiv r_iu + s_iv
\end{aligned}
$$

....

$$G_n(u,v) \equiv L_n G_{n-1} - D_y H_{n-1} \equiv p_n u + q_n v$$
$$H_n(u,v) \equiv M_n G_{n-1} + D_x H_{n-1} \equiv r_n u + s_n v,$$

we introduce v^1 as any function satisfying the equations

$$G_n\,(u^1, v^1) = 0, \; H_n(u^1, v^1) = 0.$$

It must be shown, of course, that these equations are compatible, that is, that $(p_n s_n - r_n q_n)u^1 = 0$. To do so, we note first that $p_1 s_1 - q_1 r_1 = D_x L_1 + D_y M_1 = A_1$. For induction, we assume $p_{i-1} s_{i-1} - q_{i-1} r_{i-1} = A_1 A_2 \ldots A_{i-1} (i > 1)$. Now

$$p_i u + q_i v = L_i(p_{i-1} u + q_{i-1} v) - D_y(r_{i-1} u + s_{i-1} v),$$
$$r_i u + s_i v = M_i\,(p_{i-1} u + q_{i-1} v) + D_x(r_{i-1} u + s_{i-1} v),$$

whence

$$p_i = L_i p_{i-1} - D_y r_{i-1},\; q_i = L_i q_{i-1} - D_y s_{i-1},\; r_i = M_i p_{i-1} + D_x r_{i-1}, s_i = M_i q_{i-1} + D_x s_{i-1}.$$

and

$p_i s_i - q_i r_i = (D_x L_i + D_y M_i)\,(p_{i-1} s_{i-1} - q_{i-1} r_{i-1}) = A_1 A_2 \ldots A_i$ by the induction hypothesis. It follows, then that $(p_n s_n - r_n q_n)u^1 = A_1 \ldots A_n u^1 = 0$, which establishes the compatibility of the two equations by which we determine v^1.

Defining

$$U^2(x,y) = -\int^y G_1\,(u^1(x,y_1),\ v^1(x,y_1))\,dy_1,$$

$$V^2(x,y) = -\int^y H_1(u^1(x,y_1),\ v^1(x,y_1))\,dy_1,$$

$$U^{i+1}(x,y) = (-1)^i\int^y\int^{y_1}\cdots\int^{y_2} G_i(u^1(x,y_1),\ v^1(x,y_1))\,dy_1\,dy_2\,\cdots\,dy_i,$$

$$V^{i+1}(x,y) = (-1)^i\int^y\int^{y_1}\cdots\int^{y_2} H_i(u^1(x,y_1),\ v^1(x,y_1))\,dy_1\,dy_2\,\cdots\,dy_i,$$

$$(i = 2,\ \ldots,\ n-1),$$

we see at once that $E_1(u^1,\ v^1,\ U^2) = 0$, $F_1(u^1,\ v^1,\ V^2) = 0$,

$D_y^{i-1}E_i(U^i,\ V^i,\ U^{i+1}) = (-1)^{i-1}(L_iG_{i-1} - D_yH_{i-1}) + (-1)^iG_i = 0,$

and, similarly, $D_y^{i-1}\ F_i(U^i,\ V^i,\ V^{i+1}) = 0$, $(1 < i < n)$,

and $D_y^{n-1}\ E_n(U^n,V^n) = -(L_nG_{n-1} - D_yH_{n-1}) = G_n = 0$, $D_y^{n-1}\ F_n(U^n,V^n) = 0.$

Thus,

$$E_i(U^i,V^i,U^{i+1}) = \sum_{k=0}^{i-2} y^k f_{ik}(x),\quad F_i(U^i,V^i,V^{i+1}) = \sum_{k=0}^{i-2} y^k g_{ik}(x).$$

Finally, let $F_{ik}(x)$, $G_{ik}(x)$ be such as to satisfy the system of ordinary differential equations obtained from

$$E_i\Big(\sum_{k=0}^{i-2} y^kF_{ik}(x),\ \sum_{k=0}^{i-2} y^kG_{ik}(x),\ 0\Big) = \sum_{k=0}^{i-2} f_{ik}(x)\,y^k$$

$$F_i\Big(\sum_{k=0}^{i-2} y^kF_{ik}(x),\ \sum_{k=0}^{i-2} y^kG_{ik}(x),\ 0\Big) = \sum_{k=0}^{i-2} y^k\,f_{ik}(x)$$

upon equating coefficients of the same powers of y. Then

$$u^i = U^i - \sum_{k=0}^{i-2} y^k F_{ik}(x), \quad v^i = V^i - \sum_{k=0}^{i-2} y^k G_{ik}(x) \quad \text{satisfy}$$

$E_i(u^i, v^i, u^{i+1}) = 0$, $F_i(u^i, v^i, v^{i+1}) = 0$, as desired.

4. Algebraic properties of solutions of systems of first-order equations. The special character of the canonical forms of systems of first-order linear equations for functions of two variables has far-reaching formal consequences. We consider, first, the canonical elliptic form of (2.3), an operator-matrix representation of which is

$$\begin{pmatrix} L^1(V,W) \\ M^1(V,W) \\ L^2(V,W) \\ M^2(V,W) \\ \vdots \\ L^r(V,W) \\ M^r(V,W) \end{pmatrix} =$$

$$\begin{pmatrix}
D_x+s'D_y & -s''D_y & D_y & 0 & & & & & & \\
s''D_y & D_x+s'D_y & 0 & D_y & & & & & & \\
 & & D_x+s'D_y & -s''D_y & D_y & 0 & & & & \\
 & & s''D_y & D_x+s'D_y & 0 & D_y & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & D_x+s'D_y & -s''D_y & D_y & 0 \\
 & & & & & & s''D_y & D_x+s'D_y & 0 & D_y \\
 & & & & & & & & D_x+s'D_y & -s''D_y \\
 & & & & & & & & s''D_y & D_x+s'D_y
\end{pmatrix}
\begin{pmatrix} v^1 \\ w^1 \\ v^2 \\ w^2 \\ \cdots \\ v^{r-1} \\ w^{r-1} \\ v^r \\ w^r \end{pmatrix}$$

($D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$). Representing the left-hand side of this matrix equality by K, and the right-hand side by DT, D being the square $2r \times 2r$ operator-matrix and T the $2r \times 1$ matrix of the dependent variables, we write, for short, $K = DT$. It is now a remarkable fact that $2r$ linear transformations A_j exist of $2r \times 1$ matrices into $2r \times 1$ matrices such that $A_j K = D(A_j T)$ ($j = 1, \ldots, 2r$) and that, moreover, the square matrix $(A_1 T, A_2 T, \ldots, A_{2r} T)$ is in general non-singular. Indeed, taking

$$(A_1 T, A_2 T, \ldots, A_{2r} T) = \begin{pmatrix} v^r & -w^r & v^{r-1} & -w^{r-1} & \cdots & v^1 & -w^1 \\ w^r & v^r & w^{r-1} & v^{r-1} & \cdots & w^1 & v^1 \\ 0 & 0 & v^r & -w^r & \cdots & v^2 & -w^2 \\ 0 & 0 & w^r & v^r & \cdots & w^2 & v^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & v^r & -w^r \\ 0 & 0 & 0 & 0 & \cdots & w^r & v^r \end{pmatrix}$$

both statements are immediately verified.

We shall obtain a more convenient expression of the matrix identity

(4.2) $(A_1 K, \ldots, A_{2r} K) = D(A_1 T, \ldots, A_{2r} T)$.

In terms of the standard basis (e_{ij}) ($i, j = 1, \ldots, 2r$) of the ring of square matrices of order $2r$, let us define

$\bar{e}_1 = e_{1,2r-1} + e_{2,2r}$, $\bar{e}_2 = e_{1,2r-3} + e_{2,2r-2} + e_{3,2r-1} + e_{4,2r}$ and, for $p \leq r$,

$\bar{e}_p = \bar{e}_p^{(2r)} = \sum_{q=1}^{2p} e_{q,2r-2p+q}$. $\bar{e}_p$ is a $2r \times 2r$ matrix having unity in the $(2r-2p)$th

diagonal above the principal diagonal and zero elsewhere; e_r is the identity matrix. Let us also introduce

$$i = i^{(2r)} = \sum_{q=1}^{r} (-e_{2q-1,2q} + e_{2q,2q-1}) = \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}$$

We observe

$$(4.3)\quad i^2 = -e_r,\ ie_r = e_r i = i,\ ie_p = e_p i,\ e_p e_q = e_q e_p = \begin{cases} e_{p+q-r}, & \text{if } p+q > r, \\ 0, & \text{if } p+q \leq r, \ * \end{cases}$$

so that the algebra A over the reals generated by $i, e_1, \ldots, e_r$ is commutative (and, of course, associative, since this algebra has a matrix representation.) e_r is the identity in A.

The elements of A of the form $(a + ib)e_r$ (a,b real) constitute a sub-algebra C which is obviously isomorphic to the field of complex numbers; the elements $\sum_{k=1}^{r-1} (a_k + ib_k)e_k$, of which the r-th power necessarily vanishes, comprise the radical E of A. A is, moreover, the direct sum of C with E, since any element x of A can be written uniquely as $[x_1 e_1 + \ldots + x_{r-1} e_{r-1}] + x_r e_r$, where $x_k = x_k' + ix_k''$ (x_k', x_k'' real), the bracketed sum being an element of E and the last term one of C. Finally,

---

* relations which uniquely characterise the quantitites i, e_p.

an element x of A is <u>regular</u>, i.e., has an inverse, if and only if x is not an element of the radical E. For no element of E, being a divisor of zero, can have an inverse, while, conversely, any element $y-e$ ($y \neq 0$ in C, e in E) satisfies the identity

$$(y - e)\,(y^{-1} + y^{-2}e + \dots + y^{-r}e^{r-1}) = e_r .$$

By placing the matrix interpretation upon the elements of A, it is seen that equation (4.2) can be written as

$$(4.4)\quad \sum_{p=1}^{r} (L^p + iM^p)\, e_p = \left[(D_x + s'D_y)e_r + is''D_y + e_{r-1}D_y\right] \sum_{p=1}^{r} (v^p + iw^p)e_p ,$$

a result, incidentally, whose equivalence with the original set of equations (2.3) is readily checked directly using the rules (4.3). This formula will be fundamental in later sections on elliptic systems.

Let $K(T) = DT$ be a canonical elliptic system of forms. We have shown that the one-columned matrix

$$T = \begin{pmatrix} v^1 \\ w^1 \\ \cdot \\ \cdot \\ \cdot \\ v^r \\ w^r \end{pmatrix}$$

can be augmented by the adjunction of new columns, each new column being a linear transform of T, such that the augmented matrix T^* is non-singular, assuming v^r and w^r do not both vanish, and that the new system of forms $K^*(T) = DT^*$ is equivalent * to the given system of

---

* The two sets of forms are "equivalent" in the sense that each form of one set appears also in the other, and conversely.

forms. Moreover, with $K(U) = DU$, $K^*(U) = DU^*$, the matrix U^* commutes with the matrix T^*.

Any canonical parabolic system of forms, say

$$(4.5) \qquad \begin{aligned} L^m(U) &= u^m_x + zu^m_y + u^{m+1}_y \qquad (m = 1, \ldots, t-1) \\ L^t(U) &= u^t_x + zu^t_y , \end{aligned}$$

has corresponding properties. Indeed, we verify at once the validity of

$$(4.6) \quad \begin{pmatrix} L^t & L^{t-1} & \cdots & L^2 & L^1 \\ 0 & L^t & \cdots & L^3 & L^2 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & L^t & L^{t-1} \\ 0 & 0 & \cdots & 0 & L^t \end{pmatrix} = \begin{pmatrix} D_x + zD_y & D_y & 0 & \cdots & 0 \\ 0 & D_x + zD_y & D_y & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & D_x + zD_y & D_y \\ 0 & 0 & 0 & 0 & D_x + zD_y \end{pmatrix} \begin{pmatrix} u^t & \cdots & u^2 & u^1 \\ 0 & \cdots & u^3 & u^2 \\ \vdots & & \vdots & \vdots \\ \vdots & \cdots & u^t & u^{t-1} \\ 0 & \cdots & 0 & u^t \end{pmatrix},$$

which in terms of the matrix quantities

$$c_m = c_m^{(r)} = \sum_{q=1}^{m} e_{q,r-m+q},$$

can be written

$$(4.7) \qquad \sum_{m=1}^{t} L^m c_m = \left[(D_x + zD_y)c_t + D_y c_{t-1}\right] \sum_{m=1}^{t} u^m c_m .$$

c_m is a txt matrix having unity in the $(t-m)$-th diagonal above the principal diagonal; c_t is the identity matrix. The c_m satisfy

$$(4.8) \qquad \begin{aligned} c_m c_k = c_k c_m &= c_{m+k-t}, \text{ if } m + k > t, \\ & 0, \qquad \text{ if } m + k \leq t, \end{aligned}$$

and thus generate a commutative (and associative) algebra B. The elements of B of the form ac_t (a real) consitutue a subalgebra R which is obviously isomorphic to the field of real numbers; the elements $\sum_{k=1}^{t-1} a_k c_k$ (a_k real), of which the t-th power necessarily vanishes, comprise the radical F of B. B is, moreover, the direct sum of R with F. Finally, an element of B is regular, if and only if it is not an element of the radical F; and the inverse of $y + f$ ($y \neq 0$ in R, f in F) is

$$\sum_{k=1}^{n} \frac{(-f)^{k-1}}{y^k}$$

The set of dependent variables in a canonical elliptic or parabolic system of forms will be called <u>degenerate</u>, if the variables (or variable) of highest index vanish. We can partially summarize the foregoing results in

THEOREM 4.1. Let $K(T) = DT$ be a canonical elliptic or parabolic system of expressions, D being a square matrix of differential operators as in (4.1) or (4.6), and T a one-columned matrix of the dependent variables, assumed non-degenerate. Then T can be augmented by the adjunction of new columns, each new column being a linear transform of T, such that the augmented matrix T* is non-singular, and the new system of forms $K^*(T) = DT^*$ is equivalent * to the given system of forms. Moreover, with $K(U) = DU$, $K^*(U) = DU^*$, the matrix U* commutes with the Matrix T*. If U* is a constant matrix, U* commutes also with D.

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* The two sets of forms are "equivalent" in the sense that each form of one set appears also in the other and conversely.

Any canonical system of linear expressions is an aggregation of canonical elliptic and of canonical parabolic (and hyperbolic) expressions, say $K_j(T_j) \equiv D_j T_j$, where D_j is a square matrix of differential operators, T_j an $r_j x1$ matrix of dependent variables, ($\sum r_j = n$), and may be regarded as their direct sum:

$$(4.9) \qquad K(T) = \begin{pmatrix} K_1(T_1) & & & \\ & K_2(T_2) & & \\ & & \ddots & \\ & & & K_s(T_s) \end{pmatrix}$$

$$= \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_s \end{pmatrix} \begin{pmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_s \end{pmatrix} = DT.$$

D is an nxn matrix, n being the number of equations and of dependent variables, and T an sxn matrix. Applying Theorem (4.1) individually to $T_1, T_2, \ldots, T_s$, we have the result stated in

THEOREM 4.2. Consider a canonical system of linear expressions $K(T) \equiv DT$ as presented in (4.9), assuming no T_k to be degenerate. Then the system $K(T)$ is equivalent to a new system of expressions $K^*(T) \equiv DT^*$ such that T^* is non-singular, and each column of T^* is a linear transform of a column of T. Moreover, with $K(U) = DU$, $K^*(U) = DU^*$, the matrix U^* commutes with the matrix T^*. If U^* is a constant matrix, U^* commutes also with D.

This theorem is perhaps best illustrated with the system $L(u,v) = u_x - v_y$, $M(u,v) = u_y + v_x$, which can be written $L(u,v) + iM(u,v) = (D_x + iD_y)(u + iv)$ * .

Certain formal consequences are immediate. Letting $L(u) \equiv Du$ represent a canonical system of forms with D a differentiation matrix and u a non-singular matrix of the dependent variables, we have

(4.10) $L(uv) = uL(v) + vL(u)$;

from $0 = L(uu^{-1}) = uL(u^{-1}) + u^{-1}L(u)$,

(4.11) $L(u^{-1}) = u^{-2}L(u)$;

by induction,

(4.12) $L(u^m) = mu^{m-1}L(u)$ (m a positive, negative or zero integer);

and thus, if P(x) is a rational function of x,

(4.13) $L(P(u)) = P'(u)L(u)$.

Thus, we may state

THEOREM 4.3. Let $L(u) = Du$ represent a canonical system of forms with D a differentiation matrix and u a non-singular matrix of the dependent variables. Then $L(u) = 0$, $L(v) = 0$ entail $L(uv) = 0$ and $L(u^{-1}) = 0$: i.e., the set of non-degenerate solutions of $L(u) = 0$ is a field. In particular, therefore, if P(x) is a rational function of x, $L(u) = 0$ implies $L(P(u)) = 0$.

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* where, to obtain a strict matrix interpretation, we may take

$$u + iv = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}.$$

5. Algebraic properties of systems of equations in more than two independent variables. It is of interest to inquire how far Theorem 4.2 might apply to systems of linear equations which involve more than two independent variables. We shall here consider only entirely homogeneous equations * with constant coefficients, for example,

$$(5.1) \qquad L(u) = a^k \frac{\partial}{\partial x^k} u = 0,$$

where each a^k is an nxn matrix of real constants, a^1 is non-singular **, and

$$u = \begin{pmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^n \end{pmatrix},$$

the u^q being the dependent variables. From the standpoint adopted, two questions arise: (1) When is there at least one pair of nxn matrices A, B, such that $L(Au) = BL(u)$? (2) For a given system (4.14), how many such pairs of matrices are there? Are there, in particular, n matrices $A_1, \ldots, A_n$ of the type of A such that $U = (A_1 u, A_2 u, \ldots, A_n u)$ is a non-singular matrix?

We shall answer the first question in full and the second in part. Doubtless, the second question also can be fully answered by further application of the methods employed. Of help will be

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* An equation is called entirely homogeneous, if no terms appear other than the principal part.

** as can be assured by rotation, assuming the equations to be independent.

LEMMA 5.1. Suppose every vector u satisfying

$$L(u) \equiv \sum_{k=1}^{n} a^k u_{,k} = 0 \qquad (u_{,k} = \frac{\partial}{\partial x^k})$$

also satisfies

$$M(u) = \sum_{k=1}^{n} b^k u_{,k} = 0$$

where a^k, b^k are constant nxn matrices, and a^1 is non-singular. Then there exists a matrix B such that $b^k = Ba^k$, i.e., $M = BL$.

Proof: Solutions of $L(u) = 0$ exist of the form $u = cx$, where $x = \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ x^n \end{pmatrix}$,

and c is a square constant matrix with columns designated as $c_1, \ldots, c_n$. By our assumptions,

$$(5.2) \qquad \sum_{k=1}^{n} a^k c_k = 0$$

implies

$$\sum_{k=1}^{n} b^k c_k = 0.$$

Call $(a^1)^{-1} = a$. From the preceding equations,

$$(5.3) \qquad \sum_{k=2}^{n} (b^1 a a^k - b^k) c_k = 0.$$

We can, however, select $c_2, \ldots, c_n$ so that (4.16) is not satisfied, unless all the coefficients are zero, and then determine c_1 to accord with (4.15). This would be a contradiction, whence we conclude that $b^1 a a^k - b^k = 0$ $(k = 2, \ldots, n)$, or $b^k = Ba^k$, as desired, where $B = b^1 a$.

An immediate corollary is

THEOREM 5.1. Let $L(u)$ be given as in Lemma 4.1 but with a^1 = the n-rowed identity matrix. A ncessary and sufficient condition that for a constant nxn matrix A $L(u) = 0$ implies $L(Au) = 0$ is then that A commute with each a^k.

Proof: Sufficiency is obvious. To prove the necessity, apply the preceding lemma with $b^k = a^k A$. Thus there exists a matrix B such that $a^k A = B a^k$ $(k = 1,\ldots,n)$. Taking $K = 1$ shows $A = B$, and $k = 2,\ldots,n$ give the stated commutation relations.

This theorem, in principle, answers our questions. A more specific answer than this is available, however, if we now further require that the first two terms

$$a^1 \frac{\partial}{\partial x^1} u + a^2 \frac{\partial}{\partial x^2} u$$

of $L(u)$ be in canonical form. Thereby, a^2 is, in fact, so narrowly restricted that precise characterization can be made of any matrix A that commutes with it and, further, of the matrices that commute with A. In this way, conditions can be stated for $a^3,\ldots,a^n$ which are necessary and sufficient that there exist a matrix A which commutes with $a^1 = 1$, $a^2,\ldots,a^n$. The discussion will be presented in a series of lemmas, proofs of which are here omitted.

LEMMA 5.2. Let a_j $(j = 1,\ldots,s)$ be an nxn matrix whose only non-zero elements fall in a diagonal block, say in the $(n_{j-1}+k)$-th rows, $(n_{j-1}+\ell)$-th columns $(n_0 = 0;\ 1 \leq k,\ell \leq n_j - n_{j-1};\ n_{j-1} < n_j \leq n)$ *

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* Implicitly, it has been assumed that the specified diagonal blocks for a_j, a_k $(j \neq k)$ do not overlap.

Let $K_j = (K_j^{pq})$ $(j = 1, \dots, s)$ be the n×n matrix whose only non-zero elements are

$$K_j^{n_{j-1}+k,\, n_{j-1}+k} = 1 \qquad (k = 1, \dots, n_j - n_{j-1}).$$

Then an n×n matrix A commutes with

$$a = \sum_1^s a_j$$

if and only if

(5.4) $$(K_pAK_q)a_q = a_p(K_pAK_q). \quad **$$

LEMMA 5.3. Let $U = us_n^{(n)} + c_{n-1}^{(n)}$, $V = vc_m^{(m)} + c_{m-1}^{(m)}$, and suppose M is an m×n matrix satisfying $MU = VM$. If $u \neq v$, M is then necessarily zero. If $u = v$, the elements in each diagonal*** are equal, and all elements M_{ij} for which $i-j \geqslant \text{Max}\,(1, m-n)$ vanish.

LEMMA 5.4. Let $U = u_oI^{(2n)} + u_1 i^{(2n)} + e_{n-1}^{(2n)}$, $V = v_oI^{(2m)} + v_1 i^{(2m)} + e_{m-1}^{(2m)}$, ($I^{(r)}$ is the r-rowed identity), and suppose M is a 2m×2n matrix satisfying $MU = VM$. Then $M = 0$, unless $u_o = v_o$ and $u_1 = \pm v_1$. Otherwise, writing $M = (M_{pq})$, where each M_{pq} $(p = 1, \dots, m;\ q = 1, \dots, n)$ is a 2x2 matrix of real numbers,

$$M_{p,q-1} = M_{p+1,q} \qquad (p = 1, \dots, m;\ q = 1, \dots, n)$$

** K_pAK_q is, of course, the matrix obtained from A by suppressing all elements not in rows numbered $n_{p-1} + 1, \dots, n_p$ and columns numbered $n_{q-1} + 1, \dots, n_q$.

*** running downward (upward) to the right (left).

with the convention $M_{m+1,q} = M_{p,0} = 0$. Thus, in particular $M_{pq} = 0$ for $p-q \geq \mathrm{Max}\,(1, m-n)$.

LEMMA 5.5. Let $U = uc_n^{(n)} + c_{n-1}^{(n)}$, $V = v_o + v_1 i^{(2m)} + e_{m-1}^{(2m)}$ $(v_1 \neq 0)$. If M is a $2m \times n$ matrix such that $MU = VM$, then $M = 0$.

LEMMA 5.6. Let $U = u_o + u_1 i^{(2n)} + e_{n-1}^{(2n)}$, $V = vc_m^{(m)} + c_{m-1}^{(m)}$. If M is an $m \times 2n$ matrix such that $MU = VM$, then $M = 0$.

The preceding five lemmas afford an accurate description of the matrices A that commute with a given matrix

$$a = a^2 = \begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & a_s \end{pmatrix}$$

which is the direct sum of canonical submatrices. If in particular, the canonical submatrices $a_1, \ldots, a_s$ are unlike * , A also must be of the form

$$A = \begin{pmatrix} A_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & A_s \end{pmatrix}$$

* For present purposes, we shall call two canonical matrices <u>unlike</u>, if they are distinct, except that matrices of the form $u_o I^{(2n)} + u_1 i^{(2n)} + e_{n-1}^{(2n)}$, $v_o I^{(2n)} + v_1 i^{(2n)} + e_{n-1}^{(2n)}$ will be called unlike, if either $u_o \neq v_o$ or $u_1 \neq \pm v_1$.

A_k being of the same order as a_k and of the type indicated by Lemma 5.3 or 5.4. Corresponding to

$$a_j = \begin{pmatrix} c & 1 & \\ & c & 1 \\ & & c \end{pmatrix},$$

For example, A_j is any linear combination of the three matrices

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

With

$$a_k = \begin{pmatrix} c & -d & 1 & 0 \\ d & c & 0 & 1 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix},$$

A_k is any combination of

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If, on the other hand,

$$a = a^2 = \begin{pmatrix} a_{11} & & & & & & & & \\ & \ddots & & & & & & & \\ & & a_{1s_1} & & & & & & \\ & & & a_{21} & & & & & \\ & & & & \ddots & & & & \\ & & & & & a_{2s_2} & & & \\ & & & & & & \ddots & & \\ & & & & & & & & a_{ts_t} \end{pmatrix},$$

where $a_{11},\dots,a_{1s_1}$ and $a_{21},\dots,a_{2s_2}$, etc., are sets of like matrices (with a_{1k} assumed, in addition, to be unlike any a_{2j}, etc.), any matrix A that commutes with a is of the form

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & & A_t \end{pmatrix}$$

where each A_k is of the same order as the block $\begin{pmatrix} a_{k1} & & \\ & \ddots & \\ & & a_{ks_k} \end{pmatrix}$ and of the

type indicated by Lemmas 5.3 and 5.4.

After the matrices that commute with a^2 have been determined, it would be a simple matter, in principle, to test which of them also commute with $a^3, \ldots, a^n$ in (5.1). We shall not treat this question in detail but state for the simplest case

THEOREM 5.2. Let $a, b, c, \ldots$ be a set of square matrices of order n, where a is the direct sum of unlike canonical sumatrices:

(5.12)

$$a = \begin{pmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & a_s \end{pmatrix}.$$

The linear space A of matrices that commute with a is n-dimensional. The subspace S of matrices that commute with $a, b, c, \ldots$, coincides with A, if and only if $a, b, c, \ldots$ are all contained in A.

<u>Proof</u>: The matrices that commute with a are the matrices of the form

$$\begin{pmatrix} b_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & b_s \end{pmatrix},$$

b_k being of the same order r_k as a_k, where

$$b_j = \sum_{m=1}^{r_j} b_{jm} c_m^{(r_j)} \qquad (b_{jm} \text{ real numbers}),$$

if a_j is of the parabolic type, and

$$b_k = \sum_{m=1}^{(1/2)r_k} (b'_{km} + i^{(r_k)} b''_{km}) e_m^{(r_k)} \qquad (b'_{km}, b''_{km} \text{ real}),$$

if a_k is of the elliptic type. The space A of matrices commuting with a is thus n-dimensional and contains as a subspace S those matrices that commute with $a, b, c, \ldots$. If S = A, S must contain the element

$$B = \begin{pmatrix} B_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & B_s \end{pmatrix}$$

where

$$B_j = c_{r_j - 1}^{(r_j)} ,$$

if a_j is of parabolic type, and

$$B_k = i^{(r_k)} + e_{(1/2)r_k - 1}^{(r_k)},$$

if a_k is of the elliptic type. B is, however, of the same form as a (i.e., Lemmas 5.2 to 5.6 apply to B), so that the space of matrices that commute with B is again A. The matrices $b, c, \ldots$ are, consequently,

in A.

Conversely, suppose $a, b, c, \ldots$ are contained in A. The elements of A all commute: Hence, S = A.

Theorems 5.1 and 5.2 combined with the methods of section 4 furnish a partial generalization to Theorem 4.2, namely,

THEOREM 5.3. Consider the system of linear expression

$$L(u) = \sum_{k=1}^{m} a^k u_{,k},$$

where a^1 is the n-rowed identity, $a^2, \ldots, a^n$ are nxn matrices continuous over a domain R in $x^1, \ldots, x^m$, and u is the nx1 matrix of dependent variables. In addition, a^2 is assumed to be the direct sum of unlike canonical submatrices, and, further, the set of the dependent variables associated with any of these submatrices is supposed to be non-degenerate *. We assume, finally, that $a^3, \ldots, a^m$ commute with a_2. Then the system of expressions L(u) is equivalent to the system of expressions L(U), where U is an nxn non-singular matrix each column of which is a linear transform of u. If, moreover, the nxn matrix T bears to the nx1 matrix t the same relation as U to u, then T commutes with U.

Further extensions of the theorems of section 4 are possible but will not be carried out here.

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* In the sense of section 4.

CHAPTER II

REPRESENTATIONS OF SOLUTIONS OF ELLIPTIC SYSTEMS OF EQUATIONS

1. Integral formulas. The fact that the non-degenerate solutions of a canonical elliptic system of equations form a field makes possible the construction of integral representations analogous to the Cauchy formula, for the solutions of systems of equations of the form

$$(1.1)\quad DU \equiv (D_x + iD_y + e_{r-1}(aD_x + bD_y)) \sum_{1}^{r} (u_p' + iu_p'')e_p = f \; *$$

or

$$(1.2)\quad D^*U = (D_x + iD_y + e_{r-1}(D_x a + D_y b)) \sum_{1}^{r} (u_p' + iu_p'')e_p = f \; **$$

where, in a domain R, $a = a' + ia''$, $b = b' + ib''$, and $f = \sum_{1}^{r} (f_p' + if_p'')e_p$ ($a', a'', b', b'', f_p', f_p''$ real-valued) are sufficiently smooth functions of x,y. D* will be called the adjoint of D, and D the adjoint of D*.

Only certain types of domains will be considered. These are described in the

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* The appearance in this form of $D_x + iD_y$ in place of $AD_x + BD_y$ (B/A non real) is only apparently a specialization. It is well known that, by an appropriate change of variables, the latter operator can be reduced to the former.

** It is to be understood that a differentiation operator acts upon all the factors to its right: the expression $(D_x a + D_y b)U$ thus is interpreted as $D_x(aU) + D_y(bU)$.

Definitions 1.1. Let S be a bounded domain whose boundary consists of a finite number of simple closed curves C_i $(i = 1,\ldots,N)$. It is assumed that a tangent to C_i exists at each point $z \in C_i$ $(i = 1,\ldots,N)$. If the angle $\varphi = \varphi(x,y) \equiv \varphi(z)$ $(z \in \bigcup_i C_i)$ made by Ox with the tangent to the boundary of S at z is an H-continuous * function of z, we shall say that the domain S is regular. Letting s_i represent arc length along C_i, and at points of C_i representing the angle $\varphi = \varphi(z) \equiv \phi(s_i)$ as a function of s_i, we shall say that S is regular of order k $(k \geq 1)$ if this function $\phi(s_i)$ has H-continuous k-th derivatives with respect to s_i $(i = 1,\ldots,N)$.

In a regular domain R, if U and V are any continuously differentiable hypercomplex ** functions, the identity

$$(1.3) \quad VDU + UD^*V = D^*(UV)$$

holds, and, with it, Green's formula

$$(1.4) \quad \iint_R (VDU + UD^*V)\,dxdy = -i \int_{\dot R} UV\,(dx + idy + ie_{r-1}(ady - bdx))$$

where $\dot R$ is the boundary of R. From Green's formula, the desired integral

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* A function f(z) is said to be continuous in the sense of Hölder, or H-Continuous, in a domain R, if to each $z \in R$ correspond positive numbers M, α such that $|f(z) - f(z')| < M\,|z-z'|^{\alpha}$ for all $z' \in R$. If M and α can be chosen independent of z, f(z) is called uniformly H-continuous in R.

** A number $\sum_1^n e_p(a_p + ib_p)$ $(a_p, b_p$ real) will be called hypercomplex (or hypercomplex-valued). It will be called complex (complex-valued), IN CASE $a_p + ib_p = 0$ $(p = 1,\ldots,r-1)$.

representations for the solutions of (1.1) or (1.2) will be derived through replacing V by a suitable elementary solution $V(z;z^1)$ of the adjoint system of equations.

If the domain R and the coefficients $a(x,y) \equiv a(z)$, $b(x,y) \equiv b(z)$ $(z = x + iy)$ are sufficiently regular, we shall, in fact, construct a function $V(x,y;x',y') \equiv V(z;z')$ $(z' = x' + iy')$ which is continuously differentiable for distinct z,z' in R $(z \neq z')$ and which, for z' fixed in R, satisfies the differential equations

$$DV(z;z') = 0,$$
$$D^*V(z';z) = 0,$$

and, in addition, the relations

$$\text{(1.5a)} \qquad \lim_{\rho \to 0} \int_{C_\rho} U(z)\, V(z;z')\, (dz + ie_{r-1}\,(ady - bdx)) = U(z'),$$

$$\text{(1.5b)} \qquad \lim_{\rho \to 0} \int_{C_\rho} U(z)\, V(z';z)\, (dz + ie_{r-1}\,(ady - bdx)) = -U(z'),$$

where $U(z)$ is any function continuous at z', and C_ρ is a circle of radius ρ about z'. Possession of such a $V(z;z')$, obviously, would enable us to derive an analog to Cauchy's formula from (1.4). The construction of such a function depends upon certain preliminary lemmas to which we now turn.

LEMMA 1.1. Let R be an open region bounded by a simple closed curve C which has a tangent at each point. Assume the angle φ made with Ox by this tangent is a continuous function of arc length s

along C. If $f(z)$ is a function defined and H-continuous along C:

$$\left| f(z') - f(z'') \right| < M \left| z'-z'' \right|^{\alpha} \qquad (0 < \alpha \leqslant 1),$$

then the integral

$$F(z') = \int_C f(z) \frac{dz}{z-z'}$$

is uniformly H-continuous in R.

COROLLARY: If $f(z)$ has H-continuous first derivatives $f'(z) \equiv \frac{df}{dz}$ along C, then $F(z')$ has unformly H-continuous first derivatives in R.

Proof: The corollary would be an immediate consequence, by integration by parts along C, of the assertion of the Lemma. In proving the lemma, we first shall show that $F(z')$ is uniformly bounded in R.

Let (r,θ) be polar coordinates about a point z_0 of C, the positive tangent to C at z_0 being the direction $\theta = 0$. Thus, $\varphi = 0$ at z_0. Let $z = z_0 + re^{i\theta}$ be a variable point of C. Then there exists a positive number c such that $\cos(\theta - \varphi) > \frac{1}{2}$, $|\sin\varphi| < \frac{1}{8}$, if $|z-z_0| < c$. Since C is compact, the number c may be assumed to be independent of z_0. On C, we observe $dx = \cos\varphi\, ds$, $dy = \sin\varphi\, ds$, and

$$dr = \cos\theta\, dx + \sin\theta\, dy = \cos(\theta - \varphi)\, ds.$$

Given $z' \in R$, let z_0 be a point of C which is nearest z'. If $z = z_0 + re^{i\theta}$ is, as before, a variable point of C, then by the cosine law of trigonometry,

$$|z-z'|^2 = |z'-z_0|^2 + |z-z_0|^2 - 2\,|z'-z_0|\,|z-z_0| \sin\theta.$$

For points z of C for which $|z-z_o| < c$, we know, however, that

$$|z - z_o| = \left|\int_{s_o}^{s} \cos(\theta-\varphi)\, ds\right| > (1/2)\ |s-s_o|,$$

$$|z-z_o|\ |\sin\theta| = \left|\int_{s_o}^{s} \sin\varphi\, ds\right| < (1/8)\ |s-s_o|,$$

s_o being the value of s at z_o.

Thus, for $|z-z_o| < c$,

$$|z-z'|^2 > (1/4)\ (s-s_o)^2 - (1/4)\ |z'-z_o|\ |s-s_o| + |z'-z_o|^2.$$

The right side of this inequality can be written as

$$(3/4)\ (s-s_o)^2 + |z'-z_o| \left[|z'-z_o| - \frac{|s-s_o|}{4}\right]$$

or, alternatively, as

$$\left(\frac{|s-s_o|}{2} - |z'-z_o|\right)^2 + (3/4)\ |s-s_o|\ |z'-z_o|.$$

By the first of these expressions,

$$|z-z'| \geqslant \frac{|s-s_o|}{2}$$

for $|s-s_o| \leqslant 4\ |z'-z_o|$. By the second expression,

$$|z-z'| \geqslant \frac{|s-s_o|}{2} - |z'-z_o| \geqslant \frac{|s-s_o|}{4}$$

for $c \geqslant |s-s_o| \geqslant 4\,|z'-z_o|$. In summary,

(1.6) $$|z-z'| \geqslant \frac{|s-s_o|}{4} \quad \text{for } |z - z_o| \leqslant c.$$

Since θ is continuous, and C is compact, we may assume the constant c so small that a disk of radius c or smaller about any point of

C intersects C only in one connected arc.

Using the symbol R_e to designate the set of points of R which are at a greater distance than e from the boundary C, we observed that $F(z')$ is uniformly bounded in $R_{c/2}$. Consider now a point z' in $R - R_{c/2}$. Let z_o be a point of C nearest z', and let $C_o = C_o(z_o)$ be that subarc of C which is contained within a disk of radius c about z_o. Writing

$$F(z') = f(z_o) \int_C \frac{dz}{z-z'} + \int_C \frac{f(z)-f(z_o)}{z-z'} dz = 2\pi i f(z_o) + $$

$$+ \left\{ \int_{C_o} + \int_{C-C_o} \right\} \frac{f(z)-f(z_o)}{z-z'} dz = 2\pi i f(z_o) + F_1(z') + F_2(z'),$$

we observe that $F_2(z')$ is bounded uniformly with respect to z', since $|z-z'| \geq c/2$ on $C-C_o$. $F_1(z')$ also is uniformly bounded, since

$$|F_1(z')| \leq M \int_{C_o} \frac{|z-z_o|^\alpha}{|z-z'|} |dz| \leq 8 M \int_{s_o}^{s_o+c} (s-s_o)^{\alpha-1} ds = \frac{8}{\alpha} M c^\alpha ,$$

The uniform boundedness of $F(z')$ is an immediate consequence.

To prove the uniform H-continuity of $F(z)$ in R, we need estimate the difference $F(z') - F(z'')$ only for such pairs of points z', z'' as are nearer than an arbitrary, fixed, amount, e.g., for $|z'-z''| < a_1$ $(a_1 > 0)$. This is so because of the uniform boundedness, by which there exists a constant M_1 such that $|F(z')-F(z'')| < M_1 a_1 \leq M_1 |z'-z''|$ for $|z'-z''| \geq a_1$. It is sufficient also to restrict one of the points, say z', to a fixed ring-like domain $R-R_{a_2}$, where $a_2 > a_1 > 0$. In

fact, if $z' \in R_{a_2}$, both points z', z'' are contained in $R_{a_2-a_1}$, a region whose minimum distance from C is $a_2-a_1>0$. In such a region, however, F(z) has uniformly bounded first derivatives, and, a fortiori, satisfies a uniform Hölder condition.

Thus, we shall suppose $|z'-z''|\leq a_1$ and $|z'-C|\leq a_2$ * where a_1, a_2 are arbitrary constants which will now be fixed. We shall suppose, namely, that:

(i) $a_2 \leq c/4$;

(ii) $a_1 \leq c/8$;

(iii) both a_1 and a_2 (with $a_2 > a_1$) are so small that, if z'_0, z''_0 are points on C nearest z', z'', resp., then

$$|z''_0-z'_0| \leq 2 \quad |z''-z'| \qquad **$$

(iv) if the distance between two points z_1, $z_2 \in C$ is $|z_1-z_2|<6\, a_1$, then $|s(z_1)-s(z_2)| < 2\,|z_1-z_2|$, where s(z) represents arc length along C at z.

Let us now write

$$F(z')-F(z'') = \int_C \frac{f(z)-f(z'_0)}{z-z'}\,dz - \int_C \frac{f(z)-f(z''_0)}{z-z''}\,dz$$

$$= \int_{C_0}\left[\frac{f(z)-f(z'_0)}{z-z'} - \frac{f(z)-f(z''_0)}{z-z''}\right]dz + \left\{\int_{C-C_0}\frac{f(z)-f(z'_0)}{z-z'}\,dz - \int_{C-C_0}\frac{f(z)-f(z''_0)}{z-z''}\,dz\right\} =$$

$$= F_0 + F_1$$

* By $|z'-C|$ is meant $\inf\limits_{z\in C} |z'-z|$.

** We may, for instance, let a_2 be so small that the ring like region $R-R_{2a_2}$ is simply covered by the interior normals to C. Then a_1 can be so determined that the points of any disk in $R-R_{a_2}$ of radius $a_1 < a_2$ lie on normals to C whose directions vary sufficiently little.

We shall show there exists $M_2 > 0$, independent of z', z'', such that the curly brackets F_1 are less in absolute volue than $M_2 |z'-z''|^{\alpha}$. Setting

$$G(z^*) = \int_{C-C_0} \frac{f(z)\,dz}{z-z^*}, \qquad H(z^*) = \int_{C-C_0} \frac{dz}{z-z^*},$$

we have, infact,

$$F_1 = G(z') - G(z'') + \left[f(z_0'') - f(z_0')\right] H(z'') + f(z_0')\left[H(z'') - H(z')\right].$$

Further, by (i) and (ii), the straight segment L joining z', z'' is at no point nearer to $C-C_0$ than $(5/8)c$, whence it follows that, on L, $G(z)$ and $H(z)$ are uniformly bounded and have first derivatives which also are bounded uniformly with respect to z', z''. From this, from the H-continuity of $f(z)$, and from (iii), the statement as to F_1 follows.

To complete the proof of the theorem, we must adduce a similar property for F_0. Let K be a disk about z' of radius $2\,|z'-z''|$. If K intersects C at all, $K \subset K_\rho$, where K_ρ is a disk of radius $\rho = 4\,|z'-z''|$ about z_0'. Let C_ρ be the arc of C contained in K_ρ. Obviously, $C_0 - C_\rho$ does not intersect K_ρ or, a fortiori, K, so that, by an elementary geometrical argument, $|z-z''| \geqslant (1/2)\,|z-z'|$ for $z \in C_0 - C_\rho$.

Let us write

$$F_0 = \int_{C_\rho}\left[\frac{f(z)-f(z_0')}{z-z'} - \frac{f(z)-f(z_0'')}{z-z''}\right]dz + (z'-z'')\int_{C_0-C_\rho}\frac{f(z)-f(z_0')}{(z-z')(z-z'')}\,dz$$

$$+ \left(f(z_0'') - f(z_0')\right)\int_{C_0-C_\rho}\frac{dz}{z-z''} = F_{00} + (z'-z'')F_{01} + \left(f(z_0'')-f(z_0')\right)F_{02}.$$

The arc C_ρ lies, by (ii), in $C_o = C_o(z_o')$, and, by (ii) and (iii), in $C_o(z_o'')$. From (1.6), therefore,

(1.6.1) $$|z-z'| \geqslant (1/4) \ |s-s_o'| \,, \quad |z-z''| \geqslant (1/4)\ |s-s_o''|$$

on C_ρ , where s_o', s_o'' are, resp., the values of s on C at z_o', z_o''. It follows from this and from (iv) that

$$|F_{oo}| \leqslant \left| \int_{C_\rho} \frac{f(z) - f(z_o')}{z-z'} \ dz \right| + \left| \int_{C_\rho} \frac{f(z) - f(z_o'')}{z-z''} \ dz \right|$$

$$\leqslant 8M \int_{s_o'}^{s_o'+2\rho} (s-s_o')^{\alpha-1} ds + 8M \int_{s_o''}^{s_o''+2\rho} (s-s_o'')^{\alpha-1} ds = \frac{16M}{\alpha} 2^\alpha \rho^\alpha \leqslant \frac{16M}{\alpha} 4^\alpha \, |z'-z''|^\alpha .$$

To estimate F_{01}, we recall that $|z-z''| \geqslant \frac{|z-z'|}{2}$ on $C_o - C_\rho$. From this, from (1.6.1), and from the fact that $|z-z_o'| \leqslant |s-s_o'|$, we obtain

$$|F_{01}| \leqslant 32M \int_{C_o-C_\rho} |s-s'_o|^{\alpha-2} ds \leqslant 64M \int_{s_o'+\rho}^{z_o'+2C} s^{\alpha-2} \ ds = \frac{64 . 2^{\alpha-1} . M}{1-\alpha} \ (\rho^{\alpha-1} - C^{\alpha-1})$$

whence we deduce easily the existence of a constant M_3 independent of z', z'', such that

$$|z'-z''| \, |F_{01}| \leqslant M_3 \ |z'-z''|^\alpha .$$

To conclude the proof of the lemma, it suffices to show

$$|F_{02}| \leqslant M_4 \log \ |z'-z''| \ + M_5,$$

M_4 and M_5 being constants independent of z', z''. This is, however, an obvious consequence of the inequalities

$$|F_{02}| \leqslant \int_{C_o-C_\rho} \frac{|dz|}{|z-z''|} \leqslant 2 \int_{C_o-C_\rho} \frac{|dz|}{|z-z'|} \leqslant 8 \int_{C_o-C_\rho} \frac{ds}{|s-s_o'|} \leqslant 16 \int_\rho^{2C} \frac{ds}{s} .$$

LEMMA 1.2. Let S be a regular domain. If $P(x,y) \equiv P(z)$ is a complex-valued, uniformly H-continuous function in S, the functional

$$I(P;z') = I_S(P;z') = -(1/2\pi)\iint_S \frac{P(z)}{z-z'}\,dxdy$$

defines a solution

$$w(z) = I(P;z)$$

of the equation

$$(D_x + iD_y)w(z) = P(z)$$

which has uniformly H-continuous first partial derivatives in S.

Proof: Let z_o be any point of B. Following the lines of a well-known argument * , we may write

$$W(z') = \iint_S \frac{P(z)}{z-z'}\,dxdy = P(z_o)\,G(z') + \iint_S \left[P(z) - P(z_o)\right]\frac{1}{z-z'}\,dxdy,$$

where

$$G(z') = \iint_S \frac{dxdy}{z-z'}.$$

Thus,

$$\frac{W(z') - W(z_o)}{z'-z_o} = P(z_o)\,\frac{G(z') - G(z_o)}{z'-z_o} + \iint_S \left[P(z) - P(z_o)\right]\frac{1}{z'-z_o}\left[\frac{1}{z-z'} - \frac{1}{z-z_o}\right]dxdy.$$

The second integral on the right is known to have a limit, as $z' \to z_o$, which itself is uniformly H-continuous in S ** ; G(z) we shall shortly prove to have all derivatives at any interior point of S and its first

* See E. Hopf [1] page 203

** See E. Hopf [1] Lemma 3, pp. 203-206.

derivatives $G_x(z)$, $G_y(z)$ to be uniformly H-continuous in S. It will follow, in particular, that, at $z = z_o$, $W(z)$ has first partial derivatives whose values are easily computed from the formula. Replacing z_o, an arbitrary point of S, by z', we have, in fact,

(1.7) $$D_{x'}W(z') = P(z')\, D_{x'}G(z') + \iint_S \left[P(z) - P(z')\right] \quad D_{x'}(z-z')^{-1}\, dxdy,$$

from which, and from the analogous expression for $D_{y'}W(z')$, we obtain

$$(D_{x'}+iD_{y'})W(z') = P(z')(D_{x'}+iD_{y'})G(z')+\iint_S\left[P(z) - P(z')\right]\ (D_{x'}+ iD_{y'})(z-z')^{-1}dxdy.$$

Hence, to show that $w(z) = -(1/2\pi)W(z)$ satisfies $(D_x + i\, D_y)\, w\, (z) = P(z)$, we need only to recognize that

$$(D_{x'} + iD_{y'})(z-z')^{-1} = 0 \text{ for } z \neq z'$$

and to verify, in addition, the previous assertion as to the derivatives of $G(z)$ and the equality

$$(D_{x'} + iD_{y'})G(z') = -2\pi.$$

Let K_ρ be a circular disk of radius ρ about the point z'. $G(z')$ is the limit as $\rho \to 0$ of the proper integral

$$\iint_{S-K_\rho} \frac{dxdy}{z-z'},$$

that is, by Green's formula, of

$$(1/2)\iint_{S-K_\rho}\frac{1}{z-z'}\cdot \left\{ (D_x + iD_y)\bar{z} \right\} dxdy \quad = -(1/2)\left\{\int_{\dot{S}} - \int_{\dot{K}_\rho}\right\}\frac{\bar{z}}{z-z'}\ dz,$$

where $\bar{z} = x-iy$ is the conjugate of z, and $\dot{K}_\rho$ is the boundary of K_ρ. Hence,

$$G(z') = -\pi\bar{z}' - (1/2)\int_{\dot{S}} \frac{\bar{z}}{z-z'}\ dz,$$

$\bar{z}'$ being the conjugate of z'. Each term on the right has all derivatives

for $z' \in B$, and the integral is annihilated by the operator $D_{x'} + iD_{y'}$. Hence, $G(z')$ also has all derivatives, and

$$(D_{x'} + iD_{y'})G(z') = -2\pi ,$$

as required.

We still must demonstrate the uniform H-continuity in S of the first derivatives of $G(z)$. It is enough to discuss one of them, say $G_x(z)$, since the other is determined by $(D_x + iD_y)G(z) = -2\pi$, a relation which holds for all z in S. By the foregoing,

$$D_{x'}G(z') = -\pi - (i/2) \int_{\dot{S}} \bar{z} \frac{dz}{(z-z')^2} = -\pi + (i/2) \int_{\dot{S}} \frac{d\bar{z}}{z-z'} .$$

Writing

$$dz = e^{i\varphi} |dz| ,$$

where $\varphi = \varphi(z)$ is the angle made with Ox by the tangent to the boundary at z, we have

$$d\bar{z} = e^{-2i\varphi(z)} dz \equiv h(z)\, dz.$$

Under the restrictions imposed, $h(z)$ is H-continuous for $z \in \dot{S}$. Thus, Lemma 1.1 applies to

$$\int_{\dot{S}} \frac{d\bar{z}}{z-z'} = \int_{\dot{S}} \frac{h(z)\, dz}{z-z'} ,$$

and this completes the proof.

LEMMA 1.3. If S is a regular domain of order 1, and if $P(z)$ has uniformly H-continuous first derivatives in S, then

$$w(z) = I_S(P;z)$$

has uniformly H-continuous second derivatives in S.

Proof: By Lemma 1.1, the function $w(z)$ has uniformly H-continuous

first derivatives in S. To prove the lemma, it is enough to discuss the derivatives of $w_x(z)$, hence of $W_x(z)$, where, as before,

$$W(z') = \iint_S \frac{P(z)}{z-z'}\,dxdy = \lim_{\rho\to 0} \iint_{S-K_\rho} \frac{P(z)}{z-z'}\,dxdy,$$

$K_\rho \subset S$ being a circular disk of radius ρ about the point z' of S. By Green's formula,

$$W(z') = \lim_{\rho\to 0}\left[\left\{\int_{\dot S} - \int_{\dot K_\rho}\right\} P(z)\log(z-z')\,dy - \iint_{S-K_\rho} P_x(z)\log(z-z')\,dxdy\right]$$

$$= \int_{\dot S} P(z)\log(z-z')\,dy - \iint_S P_x(z)\log(z-z')\,dxdy.$$

Hence,

$$D_{x'}W(z') = -\int_{\dot S}\frac{P(z)}{z-z'}\,dy + \iint_S \frac{P_x(z)}{z-z'}\,dxdy.$$

The area integral on the right, by Lemma 2.2, has uniformly H-continuous first partial derivatives in S. The boundary integral $J(z')$ has all derivatives with respect to x', y' for $z' \in S$, and we shall show its first derivatives, in particular, say,

$$J_{x'}(z') = \int_{\dot S}\frac{P(z)}{(z-z')^2}\,dy,$$

to be uniformly H-continuous in S. To do so, we observe that, in the notation of Definitions 2.1, $dy = \sin\varphi\,|dz|$, since on C_i $|dz|$ is equal to the element of arc ds_i. Further, $|dz| = e^{-i\varphi}\,dz$; thus,

$$P(z)\,dy = P(z)\sin\varphi\, e^{-i\varphi}\,dz \equiv j(z)\,dz,$$

where $j(z)$ is defined and has H-continuous first derivatives $j'(z) = \frac{dj(z)}{dz}$ on each curve comprising $\dot S$. We may then, integrating by parts, write

$$J_{x'}(z') = \int_{\dot{S}} \frac{P(z)dy}{(z-z')}\, 2 = \int_{\dot{S}} \frac{j(z)dz}{(z-z')} 2 = -\int_{\dot{S}} j'(z)\frac{dz}{z-z'} ,$$

Lemma 1.1 now applies to the last integral. The result is that $J_{x'}(z')$ is uniformly H-continuous in S as stated.

As final preparation for the construction and application of an elementary solution, we introduce a convergence concept into our algebra of hypercomplex quantities. Let $U = \sum_{1}^{r} e_p u_p$ be an element of this algebra, the u_p being complex-valued numbers. We then define

$$NU = |u_r| , \quad |U| = \sum_{p=1}^{r} |u_p|$$

Evidently,

$$N(U+V) \leq NU + NV, \quad |U+V| \leq U + V ,$$
$$N(UV) = NU \cdot NV, \quad |UV| \leq r\,|U||V| .$$

Since, further,

$$U^n = \left(u_r + \sum_{p=1}^{r-1} u_p e_p\right)^n \equiv (u_r + e)^n = u_r^n + \binom{n}{1} u_r^{n-1} e + \ldots + \binom{n}{r-1} u_r^{n-r+1} e^{r-1} ,$$

e^r being zero, we have also

$$N(U^n) = (NU)^n ,$$
$$|U^n| \leq C_r\, n^{r-1} \operatorname{Max}(1, |U|^{r-1}) \cdot \operatorname{Max}((NU)^n, (NU)^{n-r+1}) \qquad (n \geq r),$$

where C_r is a constant depending on r. Defining convergence in terms of the valuation $|\;|$, we can then state:

LEMMA 1.4. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with complex coefficients which converges for $|z| < c$.

If e is a nilpotent element, then the series

$$f(z+e) = \sum_{k=0}^{\infty} a_k(z+e)^k$$

also converges for $|z| < c$. The convergence is absolute-uniform for $|z| \leqslant c_0 < c$.

COROLLARY: If, for z in some open set S, w(z) is a continuously differentiable hypercomplex function such that

$$N(w(z)) \leqslant c_0 < c,$$

then in S

$$df(w(z)) = f'(w(z))\, dw(z),$$

where $f'(z) = \frac{d}{dz} f(z)$.

<u>Proof</u>: Let

$$f_{nm}(z) = \sum_{m}^{n} a_k z^k.$$

By the foregoing rules for the new valuation, there is a constant C depending on e such that (for $m \geqslant r$)

$$\left|f_{nm}(z+e)\right| \leqslant C \operatorname{Max}\left\{\sum_{m}^{n} k^{r-1}|a_k|\,|z^k|,\ \sum_{m}^{n} k^{r-1}\,|a_k|\,|z^{k-r+1}|\right\}$$

$$\leqslant 2\, C \operatorname{Max}(1, |z|^{r-1}) \sum_{m}^{n} k(k-1)\ldots(k-r+1)\,|a_k|\,|z|^{k-r+1}.$$

Lemma 1.4 now follows from the absolute-uniform convergence for $|z| \leqslant c_0 < c$ of all the series (in particular of the (r-1)st) obtained successively from f(z) by termwise differentiation. The corollary also is proved by termwise differentiation.

The rules governing the manipulation of ordinary convergent power series apply, thus, to power series in hypercomplex numbers. A well known rearrangement theorem, in particular, gives us

LEMMA 1.5 If $f(z)$ represents a convergent power series, then

$$f(z+e) = \sum_{0}^{r-1} \frac{1}{k!} e^k f^{(k)}(z),$$

where $f^{(k)}(z) = \frac{d^k}{dz^k} f(z)$.

Proof of this result depends on the fact that $e^k = 0$ $(k = r, r+1, \ldots)$.

We can now, making use of the preceding lemmas, construct a suitable elementary solution of the equation $DV = 0$, assuming $a(x,y) \equiv a(z)$, $b(x,y) \equiv b(z)$ to be uniformly H-continous in a regular domain R. With the notation

$$I(P;z') = -(1/2\pi) \iint_R \frac{P(z)}{z-z'} \, dxdy,$$

we begin by defining

$$(1.8) \quad t_r(z) = z$$
$$t_p(z) = I\left(-(aD_x + bD_y)t_{p+1}(z);z\right) \qquad (p = 1, \ldots, r-1).$$

By Lemma 1.2, each $t_p(z)$ has uniformly H-continuous first partial derivatives in R, and the hypercomplex function

$$t(z) = \sum_{1}^{r} e_p t_p(z) \equiv z + T(z) \qquad (T(z) \text{ nilpotent})$$

satisfies

$$(1.9) \quad Dt \equiv (D_x + iD_y + e_{r-1}(aD_x + bD_y))t = 0.$$

$t(z)$ will be called a generating solution of $Dt = 0$. We observe that

$$D(t(z) - t_o)^k = 0 \qquad (t_o = \text{hypercomplex constant}),$$

and, more generally,

LEMMA 1.6. To any complex-valued function $f(z)$, regular and analytic in a subdomain $S \subset R$, corresponds a hypercomplex valued function

$$F(z) \equiv f(t(z)) \equiv \sum_{0}^{r-1} \frac{1}{k!} f^{(k)}(z) (T(z))^k$$

which at each point of S satisfies

$$dF(z) = f'(t(z))\, dt(z),$$

and, hence, in particular, the equation

$$DF(z) = 0.$$

Proof: In the neighborhood of an arbitrary point z_0 of S, $f(z)$ can be represented as a convergent power series, say

$$f(z) = s(z;z_0) = \sum_{0}^{\infty} a_k(z-z_0)^k.$$

By Lemma 1.4, the power series $s(t(z);z_0)$ converges in the neighborhood of z_0, and Lemma 1.5 then shows

$$s(t(z);z_0) = \sum_{0}^{r-1} \frac{1}{k!} s^{(k)}(z;z_0) (T(z))^k$$

$$= \sum_{0}^{r-1} \frac{1}{k!} f^{(k)}(z) (T(z))^k.$$

The differentiation rule, and, hence, the fact that the right side of the last equation is annihilated by D, are proved, finally, by applying d termwise to the series

$$s(t(z);z_0) = \sum_{0}^{\infty} a_k(t(z) - z_0)^k,$$

an operation which is justified by the corollary to Lemma 1.4.

Equation (1.9) may also be written as

$$(1+ e_{r-1}a)t_x + i(1 - ie_{r-1}b)t_y = 0,$$

and from this the identity

(1.10) $$\frac{t_x}{1-ie_{r-1}b}\left[dx + idy + ie_{r-1}(ady-bdx)\right] = t_x dx + t_y dy$$

is obvious. Now we define

(1.11) $$V(z;z') = (2\pi i)^{-1}\,\frac{t_x(z')}{1-ie_{r-1}b(z')}\cdot\frac{1}{t(z)-t(z')}.$$

This is the elementary solution of $DV = 0$ which we shall use in Green's formula. Its important properties are summarized in

THEOREM 1.1. If $a(z)$, $b(z)$ are complex-valued, uniformly H-continuous functions in a regular domain R, let

(1.12) $$t(z) = z + \sum_{1}^{r-1} e_p t_p(z)$$

be a solution of equation (1.9) which possesses uniformly H-continuous first derivatives in R. Such a solution exists. Defining $V(z;z')$ as in (1.11), we then have

(1.13) $DV(z;z') = 0.$

Further, both the relations

$$\lim_{\rho\to 0}\int_{C_\rho} U(z)\,V(z;z')\,(dx + idy + ie_{r-1}(ady-bdx)) = U(z'),$$

(1.5)

$$\lim_{\rho\to 0}\int_{C_\rho} U(z)\,V(z';z)\,(dx + idy + ie_{r-1}(ady-bdx)) = -U(z'),$$

C_ρ being a circle of radius ρ about the fixed point z' of R, are valid for any hypercomplex function $U(z)$ which is continuous at z'.

If $a(z)$, $b(z)$ possess uniformly H-continuous first derivatives in R, and if, in addition, R is regular of order 1, we may assume $t(z)$ to have uniformly H-cotinuous second derivatives in R. In this case,

(1.13*) $D^*V(z';z) = 0.$

Proof: The existence of a generating solution $t(z)$ and the continuity properties of its derivatives have already been shown. We have only to verify (1.5) and 1.13*). The second integral in (1.5) can be written

$$-(2\pi i)^{-1}\, U(z') \int_{C_\rho} \frac{dt(z)}{t(z)-t(z')} - (2\pi i)^{-1} \int_{C_\rho} (U(z)-U(z'))\, \frac{t_x(z)}{1-ie_{r-1}b(z)}\, \frac{(dz+ie_{r-1}(ady-bdx))}{(t(z)-t(z'))}$$

$$= -U(z')\, I_1 - (2\pi i)^{-1}\, I_2.$$

To evaluate I_1, we may introduce, in accordance with Lemma 1.6, the function

$$f(z) \equiv \log(t(z)-t(z')) = \log(z-z') + \sum_1^{r-1} \frac{1}{k!} \frac{(T(z)-T(z'))^k}{(z-z')^k}.$$

Obviously,

$$df(z) = d\log(z-z') + d\sum_1^{r-1} \frac{1}{k!} \left(\frac{T(z)-T(z')}{z-z'}\right)^k,$$

whence we see that

$$\int_{C_\rho} df(z) = 2\pi i.$$

The integral on the left is, however, $2\pi i I_1$, since

$$df(z) = \frac{dt(z)}{t(z)-t(z')},$$

a relation which follows from Lemma 1.6.

To prove that I_2 tends to zero with ρ, we first observe that, if w is a complex number and e nilpotent, then

$$(w-e)^{-1} = \sum_1^r w^{-k} e^{k-1}.$$

Hence,

$$(t(z)-t(z'))^{-1} = (z-z')^{-1} \sum_1^r (-1)^{k-1} \left(\frac{T(z)-T(z')}{z-z'}\right)^{k-1}$$

It follows, since $T(z)$ has uniformly bounded first derivatives in R,

that on C_ρ

$$\left|(t(z) - t(z'))^{-1}\right| \leqslant M_1 / |z-z'| = M_1/\rho ,$$

where M_1 is a suitable constant independent of ρ. Similarly,

$$\left|\frac{t_x(z)}{1 - ie_{r-1}b(z)}\right| \leqslant M_2, \quad \left|dz + ie_{r-1}(ady-bdx)\right| \leqslant M_3 \ ds,$$

M_2, M_3 being constants independent of ρ, and ds the element of arc length on C_ρ. Now let $\emptyset(\rho)$ be a modulus of continuity for $U(z)$ at z':

$$\left|U(z) - U(z')\right| \leqslant \emptyset(\rho)$$

for $|z-z'| \leqslant \rho$ with $\lim_{\rho \to 0} \emptyset(\rho) = 0$. Evidently,

$$\left|I_2\right| \leqslant M_1 M_2 M_3 \emptyset(\rho) \int_{C_\rho} \frac{ds}{\rho} = 2\pi M_1 M_2 M_3 \emptyset(\rho).$$

If follows that I_2 does tend to zero with ρ, as stated, and thus the second relation (1.5) is established.

To prove the other of these relations, let us write the first integral in (1.5) as

$$- \int_{C_\rho} U(z)\, V(z';z)\, (dx + idy + ie_{r-1}(ady-bdx))$$

$$+ \int_{C_\rho} U(z)\, (V(z';z) + V(z;z'))(dx + idy + ie_{r-1}(ady-bdx)) = J_1 + J_2.$$

We have already shown that J_1 tends to $U(z')$ as $\rho \to 0$. What remains is to prove that $J_2 \to 0$. First we note that the function

$$g(z) = \frac{t_x(z)}{1 - ie_{r-1}b(z)}$$

is uniformly H-continuous in R:

$$\left|g(z) - g(z')\right| < K\,|z-z'|^{\,h} \qquad (0<h\leqslant 1,\;\; K = \text{const.}).$$

Hence, on C_ρ,

$$\left|V(z';z) + V(z;z')\right| = \left|(2\pi i)^{-1}\,(t(z) - t(z'))^{-1}\,(g(z) - g(z'))\right|$$

$$\leqslant K_1\left|(z-z')\right|^{h-1} = K_1\,\rho^{\,h-1},$$

where K_1 is a constant independent of ρ. Also,

$$|U(z)| \leqslant K_2$$

in a neighborhood of z', say for $|z-z'| \leqslant \rho_0$. For $\rho \leqslant \rho_0$, it follows, therefore, that

$$|J_2| \leqslant \frac{K_1K_2M_3}{\rho^{1-h}}\int_{C_\rho} ds = 2\pi K_1K_2M_3\,\rho^h \to 0$$

as $\rho \to 0$, as asserted.

For such $t(z)$ as possess continuous second derivatives in R, let us, finally, verify equation (1.13*). By (1.11),

$$DV(z';z) = (1/2\pi i)\;(t'-t)^{-1}\;D\left(\frac{t_x}{1-ie_{r-1}b}\right)$$

$$= (1/2\pi i)\quad (t'-t)^{-1}\left\{\frac{Dt_x}{1-ie_{r-1}b} + \frac{ie_{r-1}\,t_x\,Db}{(1-ie_{r-1}b)^2}\right\},$$

where we have put $t = t(z)$, $t' = t(z')$, etc. We observe, further, that from $Dt = 0$ follows

$$0 = D_xDt = D_x(D_x + iD_y + e_{r-1}(aD_x + bD_y))t = (D_x + iD_y + e_{r-1}(aD_x + bD_y))D_x$$

$$+ e_{r-1}(a_xD_x + b_xD_y)\,t = Dt_x + e_{r-1}(a_xt_x + b_xt_y),$$

and, in addition,

$$(1 + e_{r-1}a)t_x + (i + e_{r-1}b)t_y = 0;$$

the last two results together give

$$Dt_x = -e_{r-1}\, t_x \left[a_x + \frac{ib_x(1+e_{r-1}a)}{1-ie_{r-1}b} \right].$$

Hence,

$$DV(z';z) = (2\pi i)^{-1} \quad (t'-t)^{-1}\; e_{r-1}\; \frac{t_x}{1-ie_{r-1}b} \left[-a_x + \frac{-ib_x(1+e_{r-1}a) + iDb}{1-ie_{r-1}b} \right]$$

$$= -V(z';z)\;(a_x + b_y).$$

Equation (1.13*) now follows from the fact that

$$D^* = D + e_{r-1}(a_x + b_y).$$

Theorem 1.1 and Green's formula, applied to a domain consisting of the points of R outside a small circle about the fixed point z', now give us

THEOREM 1.2. If $a(z)$, $b(z)$ have uniformly H-continuous first derivatives in a regular domain R, let $t(z)$ be a function of the type (1.12) as described in Theorem 1.1, and let $V(z;z')$ be defined from (1.11). Then for any hypercomplex-valued function $U(z)$ of class C' in R, there is an integral representation

$$U(z') = \int_{\dot{R}} U(z)\; V(z;z')\; (dx + idy + ie_{r-1}(ady-bdx)) - i\iint_R V(z;z')\; D^*U(z)\; dxdy.$$

If R is regular of order 1, $t(z)$ may be assumed to have H-continuous second derivatives in R, and in this case the alternative representation

$$U(z') = -\int_{\dot{R}} U(z)\; V(z';z)\; (dx + idy + ie_{r-1}(ady-bdx)) + i\iint_R V(z';z)\; DU(z)\; dxdy$$

also is valid.

Partial converses to these statements are provided by the combined assertions of the two following theorems:

THEOREM 1.3. Assume $a(z)$, $b(z)$ have uniformly H-continuous first derivatives in the regular domain R. If $f(z)$ is a hypercomplex-valued function defined and integrable on the boundary $\dot{R}$ of R, then the integrals

$$I(z') = \int_{\dot{R}} f(z)\ V(z;z')\ (dx + idy + ie_{r-1}(ady-bdx)),$$

$$J(z') = \int_{\dot{R}} f(z)\ V(z';z)\ (dx + idy + ie_{r-1}(ady-bdx))$$

represent functions with H-cotinuous first derivatives in R such that $D^*I(z) = 0$, and $DJ(z) = 0$.

Proof, in accordance with Theorem 1.1, is by differentiating under the integral signs.

THEOREM 1.4. Let R be a regular region of order 1 in which $a(z)$, $b(z)$ are assumed to have uniformly H-cotinuous first partial derivatives. Let $j(z)$ be a hypercomplex-valued function defined and having uniformly H-continuous first partial derivatives in R. If $V(z;z')$ is defined as in Theorem 1.1, the integrals

$$S(z') = \iint_R j(z)\ V(z;z')\ dxdy,$$

$$T(z') = \iint_R j(z)\ V(z';z)\ dxdy$$

have H-continuous first derivatives in R and in R satisfy the equations

$$(1.14)\quad D^*S(z) = ij(z), \quad DT(z) = -ij(z).$$

<u>Proof:</u> Reasoning as in the proof of Lemma 1.2, we can easily show that

$$T_{x'}(z') = j(z')\, D_{x'} \iint_R V(z';z)\,dxdy + \iint_R (j(z) - j(z'))\, D_{x'}V(z';z)\,dxdy$$

and that the second integral on the right satisfies a uniform H-condition in R. We must also show that

$$H(z') = \iint_R V(z';z)\,dxdy$$

has H-continuous first derivatives in R and that

(1.15) $DH(z) = -i.$

To facilitate the discussion of $H(z)$, we shall introduce a new function

$$s(z) = (1/2)\bar{z} + \sum_1^{r-1} e_p s_p(z)$$

which is to have uniformly H-continuous first derivatives in R and to satisfy there the equation

(1.16) $Ds(z) = 1,$

that is,

(1.16_p) $(D_x + iD_y)s_p(z) + (aD_x + bD_y)s_{p+1}(z) = 0$ $(p = 1,\ldots,r-1;\ s_r(z) = \frac{\bar{z}}{2})$.

Specifically, using again the notation

$$I(P;z') = -(1/2\pi) \iint_R \frac{P(z)}{z-z'}\,dxdy,$$

we define

$$s_r(z) = (1/2)\bar{z}$$

$$s_p(z) = I(-(aD_x+bD_y)s_{p+1};\, z) \quad (p = 1,\ldots, r-1).$$

It is clear from Lemma 1.2 that each $s_p(z)$ has uniformly H-continuous first derivatives in R and that equations (1.16) are satisfied.

If K_ρ is a disk of radius ρ about $z' \in R$, then $H(z')$ is the limit as $\rho \to 0$ of the proper integral

$$\iint_{R-K_\rho} V(z';z)\, dxdy = \iint_{R-K_\rho} V(z';z)\, Ds(z)\, dxdy$$

or, by Green's formula, of

$$-i\left\{\int_{\dot{R}} - \int_{\dot{K}_\rho}\right\} V(z';z)\, s(z)\, (dz + ie_{r-1}(ady-bdx)),$$

since $D^*V(z';z) = 0$ in $R-K_\rho$ $(\rho > 0)$. Theorem 1.1 thus gives us

$$H(z') = -is(z') - i\int_{\dot{R}} V(z';z)\, s(z)\, (dz + ie_{r-1}(ady-bdz)),$$

whence we observe that $H(z)$ has H-continuous first derivatives in the interior of R and that, moreover, the second equation of (1.14) is justified. The first can be proved in a similar fashion.

We shall later have need also of

THEOREM 1.5. Let $a(z)$, $b(z)$ have uniformly H-continuous first derivatives in a regular domain R of order one, and let $V(z;z')$ be defined from (1.11). Let C be a simple, closed curve contained in R having continuous curvature, and let $U(z)$ be defined and piecewise continuous on C. If $U(z)$ is H-continuous at a point z_0 of C, the integral

$$J(z') = \int_C V(z';z)\, U(z)\, (dx + idy + i\bar{e}_{r-1}(ady-bdx))$$

satisfies the relation

$$\lim_{z' \to z_0} J(z') = J(z_0) - (1/2)U(z_0),$$

as z' tends to z_0 from the interior of C.

Proof: By foregoing reasoning,

$$J(z') = -(2\pi i)^{-1} \int_C U(z) \frac{dt(z)}{t(z) - t(z')} = -(2\pi i)^{-1} \int_C U(z) \, d\log(t(z) - t(z'))$$

$$= -(2\pi i)^{-1} \int_C U(z) \, d\log(z-z') - (2\pi i)^{-1} \int_C (U(z)-U(z')) \, d \sum_1^{r-1} \left(\frac{T(z)-T(z')}{z-z'}\right)^k.$$

The second integral on the right is continuous on the boundary because of the Hölder condition supposed for $U(z)$, while the first integral satisfies

$$\lim_{z' \to z_0} (2\pi i)^{-1} \int_C U(z) \, d\log(z-z') = (2\pi i)^{-1} \int_C U(z) \, d\log(z-z_0) + (1/2)U(z_0) \text{ * }.$$

2. Representation of the solutions of homogeneous systems of equations. With the aid of Cauchy's integral formula, a simple representation can be obtained for sufficiently regular solutions of the system of equations

(2.1) $DU = (D_x + iD_y + ie_{r-1}(a(z)D_x + b(z)D_y))U(z) = 0,$

where $a(z)$, $b(z)$ are complex-valued functions possessing uniformly H-continuous first derivatives in a domain R. We shall show, namely, that the manifold of solutions of $DU = 0$ regular in the neighborhood of a point z_0 of R is the manifold of all convergent power series

(2.2) $\sum_{n=0}^{\infty} c_n (t(z) - t(z_0))^n$

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* The integral on the right converges absolutely. For proof of this formula, see R. Courant [1], pp. 306-312.

with hypercomplex constant coefficients c_n, where

$$(2.3) \quad t(z) = z + \sum_{1}^{n-1} e_p t_p(z) \equiv z + T(z) \qquad (T(z) \text{ nilpotent})$$

is a function possessing uniformly H-continuous second derivatives in a neighborhood of z_o and satisfying the equation

$$Dt(z) = 0.$$

We have previously called $t(z)$ a generating solution of $Dt = 0$. Convergence of the series is defined with respect to the metric introduced in the preceding section. The existence inside a subdomain R_o of R, which is regular of order 1, of a function of the type (2.3) with the indicated properties is guaranteed by Theorem 1.1.

Let $U(z)$ be a solution of (2.1) which is of class C' in R. Let C be a circle about z_o which with its interior is contained inside R_o. By the second formula of Theorem 1.2, and by (1.10), (1.11), the value of U at any point z' interior to C is

$$U(z') = (2\pi i)^{-1} \int_C U(z) \frac{dt(z)}{t(z) - t(z')} .$$

Employing the expansion

$$\frac{1}{t(z) - (t(z')} = \frac{1}{t(z) - t(z_o)} \sum_{k=0}^{n} \left(\frac{t(z') - t(z_o)}{t(z) - t(z_o)}\right)^k + \frac{1}{t(z)-t(z')}\left(\frac{t(z')-t(z_o)}{t(z)-t(z_o)}\right)^{n+1}$$

we see that, at the points of any disk K about z_o which is inside C, $U(z)$ is, indeed, represented by a series of the form (2.2), its coefficients defined by

$$c_k = (2\pi i)^{-1} \int_C U(z) \, (t(z) - t(z_o))^{-k-1} \, dt(z),$$

provided that for $z' \in K$ the remainder

$$(2\pi i)^{-1} (t(z') - t(z_o))^{n+1} \int_C \frac{u(z)\, dt(z)}{(t(z) - t(z_o))^{n+1}\,(t(z) - t(z'))}$$

tends uniformly to zero. This can be proved in much the same way as in the classical case, and further details will be omitted.

Attention is directed to Lemma 1.5 by which the series expansion (2.2) for an arbitrary regular solution of the system of equations (2.1) can be written as

$$U(z) = \sum_{1}^{r} e_p U_p(z) = \sum_{p=1}^{r} \sum_{k=0}^{r-1} \frac{1}{k!} (T(z) - T(z_o))^k f_p^{(k)}(z),$$

the $f_p(z)$ being functions analytic at z_o. Conversely, such an expression defines a solution $U(z)$ of (2.1).

3. Uniqueness of the solution of a Cauchy problem. An extension of a theorem of Carleman. In a well known paper, T. Carleman [1] has considered systems of linear equations of mixed elliptic and hyperbolic type of the form

$$(D_x + (A'_p + iA''_p)D_y)(u'_p + iu''_p) = \sum_{q=1}^{m} (c_{pq1}u'_q + c_{pq2}u''_q) + \sum_{q=m+1}^{n} c_{pq}u_q$$

$$(D_x + A_sD_y)u_s = \sum_{q=1}^{m} (c_{sq1}u'_q + c_{sq2}u''_q) + \sum_{q=m+1}^{n} c_{sq}u_q$$

$$(p = 1, \dots, m;\ s = m+1, \dots, n),$$

the coefficients

$$(3.1_A) \qquad A_p(x,y) = A'_p + iA''_p \qquad (A'_p,\ A''_p \text{ real};\ A''_p \neq 0;\ p = 1,\dots,m),$$
$$A_s(x,y) \qquad (A_s \text{ real};\ s = m+1,\dots,n)$$

being of class C", and the remaining coefficients

$$(3.1_c) \qquad c_{pqr} = c'_{pqr} + ic''_{pqr} \qquad (c'_{pqr},\ c''_{pqr} \text{ real}),$$
$$c_{jq} \qquad (c_{jq} \text{ real})$$

continuous, inside a semicircle

D: $x^2 + y^2 \leqslant d^2,\ x>0.$

The c_{pqs}, A_p, and the first and second derivatives of the A_p are assumed to be uniformly bounded in D *. Carleman showed that, if a continuously dif-

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* Carelman does not state this explicity, but the assumption is certainly used in the proof that $v<0$ (p.5) and in the definition of L, p. 6. This assumption may also be implicit in Carleman's assertion (equations (12), p. 4) as to the existence in D of continuously differentiable solutions u(x,y), v(x,y) of the system

$$(D_x + A_pD_y)(u + iv) = 0$$

of the form

$$(\Lambda) \quad \begin{aligned} u &= A''_p(x', y')(x-x') + A(x-x')^2 + 2B(x-x')(y-y') + C(y-y')^2 + \mathcal{E}((x-x')^2 + (y-y')^2) \\ v &= y-y' - A'_p(x',y')(x-x') + A_1(x-x')^2 + 2B_1(x-x')(y-y') + C_1(y-y')^2 + \mathcal{E}_1((x-x')^2 + (y-y')^2) \end{aligned}$$

ferentiable solution of this system vanishes on the y-axis:

$$u_p(0,y) = 0 \text{ for } |y| < d \qquad (u_p = u_p' + u_p'' \text{ for } p \leq m;\ p = 1,\dots,n),$$

there exists a radius d_o such that

$$u_p(x,y) = 0 \quad (p = 1,\dots,n)$$

in the semi-circle $x^2 + y^2 \leq d_o^2,\ x > 0$.

The importance of this theorem lies, of course, in how very little is required of the coefficients in the equations. * The results previously known had demanded analyticity of these coefficients. Previous results, on the other hand, had applied to systems of any type, while Carleman's theorem excludes all systems of partly parabolic, and some systems of partly elliptic, type. We shall show how, by Carleman's own methods, and with the aid of the results of section 1, his theorem can be extended to all systems of hyperbolic and elliptic type. mixed

We begin with the totally elliptic system of linear equations

$$(3.2) \quad (D_x + A_p D_y + e_{p,r_p-1} D_y) U_p(x,y) = \sum_{q=1}^{m} (c_{pq1}\, u_{pq}' + c_{pq2}\, u_{pq}'') \equiv G_p(U) \quad (p = 1,\dots,m)$$

where $(x', y') \in D$, and A, B, C, A_1, B_1, C_1 are functions of x', y', and ε, ε_1 functions of x, y, x', y' which tend to zero with $(x-x')^2 + (y-y')^2$ uniformly in a certain neighborhood of $x=y=x'=y'=0$. Carleman does not say what kind of construction he had in mind to produce functions of the type $(\hat{\ })$. In Chapter III, section 2 will be presented a method of construction under the condition that the A_p are defined and of class C''' in the entire circle K: $x^2 + y^2 \leq d^2$. To assure this condition, it is, however, enough, to require that the A_p and their first, second, and third derivatives be defined and uniformly continuous in the semi-circle D, as the domain of A_p, as a function of class C''', could then be extended, first to the y-axis, next to all of K.

* See the comments of Petrovskiĭ [1], pp. 4-5.

made up of subsystems whose principal parts are in canonical form. $2r_p$ represents the number of real dependent variables (and of equations for real quantities) in the principal part of the p-th subsystem, and

$$1, e_{p1}, \dots, e_{pr_p} = 1$$

the standard basis of the algebra associated with it. The coefficients (3.1_A) and (3.1_C) are defined , the first having uniformly continuous first, second, and third derivatives, the second being uniformly continuous, in D. We suppose also that a solution

$$(3.1_U) \quad U_p = \sum_{q=1}^{r_p} e_{pq}(u'_{pq} + iu''_{pq}) = \sum_{q=1}^{r_p} e_{pq}u_{pq}$$

is known which is uniformly bounded and of class C' in D and satisfied

$$(3.3) \quad U_p(0,y) = 0 \text{ for } |y| < d \qquad (p = 1,\dots,m)$$

and shall show there exists a radius d_o such that

$$U_p(x,y) = 0 \qquad (p = 1,\dots,m)$$

in the semi-circle $x^2 + y^2 \leqslant d_o^2$, $x > 0$.

Following Carleman, we introduce new variables through

$$U_p = e^{t(x + y^2 - cx^2)} V_p \quad (c,t \text{ real, positive})$$

in terms of which equations (3.2) can be written

$$(D_x + A_pD_y + e_{p,r_p-1}D_y) V_p + t(1-2cx+2A_py+2e_{p,r_p-1}y)V_p = G_p(V),$$

or, alternatively

$$(3.4) \quad (D_x + D_yA_p + e_{p,r_p-1}D_y)V_p + t(1-2cx+2A_py+2e_{p,r_p-1}y)V_p$$

$$= G_p(V) + A_{p,y}V_p \equiv H_p(V).$$

Let T_k be the hyperbolic arc defined by

$$x + y^2 - cx^2 = k^2$$

$$0 < x \leqslant \frac{1}{2c} - \sqrt{\frac{1}{4c^2} - \frac{k^2}{c}} \qquad k < 1/2\sqrt{c},$$

and let D_k be the domain bounded by T_k and the y-axis. Over D_k, we shall, like Carleman, apply Green's formula to the expression (3.4) using for this purpose a suitable singular solution of the equation

(3.5) $(D_x + A_p D_y + e_{p,r_p-1} D_y) W_p = t(1-2cx + 2A_p y + 2e_{p,r_p-1} y) W_p = 0.$

It is not difficult to produce a general representation for the solutions of equations (3.5) from which the specific singular solution desired for use in Green's formula can be obtained. Set

$$Z_p = e^{-t(x+y^2-cx^2)} W_p.$$

Equations (3.5) become

(3.6) $(D_x + A_p D_y + e_{p,r_p-1} D_y) Z_p = 0.$

We may suppose * that in D the auxil iary equations

(3.7) $(D_x + A_p D_y)(u+iv) = 0$

have a solution

$$w(z) = w(x,y) = u(x,y) + iv(x,y)$$

which possesses continuous and uniformly bounded first, second, and third derivatives and which is such that not all first derivatives u_x, u_y, v_x, v_y simultaneously vanish at any point of the closure of D. Then Taylor's Theorem with remainder, applied in a neighborhood of an arbitrary point $z' = x'+iy'$

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* See the first footnote of this section and the end of section 2, Chapter III.

of D, shows that

$$(3.8)\qquad \frac{A_p(z')}{iw_x(z')}\,(w(z) - w(z')) \equiv X(z;z') + iY(z;z')$$

is a solution of equations (3.7) of type (A). Now we change independent variables by

$$X = X(z;z'),\quad Y = Y(z;z').$$

Since

$$D_x + A_pD_y = \left[(D_x + A_pD_y)X\right]D_X + \left[(D_x + A_pD_y)Y\right]D_Y$$

($D_X = \frac{\partial}{\partial X}$, etc.), we have from (3.7) and (3.8) that

$$(D_x + A_pD_y)Z = \left[(D_x + A_pD_y)X\right](D_X + iD_Y)Z,$$

and, hence, that

$$(3.9)\quad (D_x+A_pD_y + e_{p,r_p-1}D_y)Z_p = \left[(D_x+A_pD_y)X\right](D_X+iD_Y+e_{p,r_p-1}(aD_X+bD_Y))Z_p,$$

where

$$a = X_y/(D_x+A_pD_y)X,\qquad b = Y_y/(D_x+A_pD_y)X. \;*$$

Let $s(X+iY) \equiv S(z;z')$ be a generating solution of the homogeneous equation obtained by setting the right side of (3.9) equal to zero. (The designation $t(x+iy)$ was used in section 1). As we have shown in section 1 the most general solution of this equation (i.e., of equation (3.6)) is an analytic function **

$$F(s(X+iY)) = F(S(z;z'))$$

of the generating solution; in consequence, the most general solution of

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* We observe that, by (3.8), (3.7), $(D_x+A_pD_y)X = 0$ implies $X_x = X_y = Y_x = Y_y = 0$, hence, that $w_x = w_y = 0$, an equality that is excluded in the closure of D.

** more exactly, a sum of such functions with hypercomplex coefficients

(3.5) is a function of the form

(3.10) $W_p = e^{t(x+y^2-cx^2)} F(S(z;z'))$.

As the first step in suitably choosing F, we recall that X and Y are expressible in terms of x,y by equations of the form (A) of a preceding footnote. By a result of Carleman (pp.[illegible]), it then follows that there exist numbers ρ_1, c, independent of p, such that

$$x + y^2 - cx^2 = C_0 + C_1 X + C_2 Y + 2C_3 XY + C_4(X^2-Y^2) + (X^2+Y^2)R(X,Y)$$

for

(3.11) $|z| < \rho_1, \quad x > 0,$

where the C_q are complex-valued constant, and

(.12) $R < 0$

in the closure of the domain (3.11).

The second step indefining F is to introduce

$$F_1(S(z;z')) = F_1(s(X+iY))$$
$$= e^{-t(C_o + (C_1-iC_2)s(X+iY) + (C_4-iC_3)(s(X+iY))^2} (s(X+iY))^{-1}.$$

By definition of $s(X+iY)$, see (1.8),

$$e^{t(x+y^2-x^2)} \; F_1(S(z;z')) = e^{t(X^2+Y^2)R(X,Y)} e^{itQ(z;z')} e^{tN(z;z')} (S(z;z'))^{-1},$$

where Q is a real-valued function and N nilpotent * . Now we can define

$$\Gamma(S(z;z')) = (2\pi i)^{-1} e^{-iQ(z';z')t - N(z';z')t} S_y(z';z') F_1(S(z;z')).$$

It follows, in view of (3.12) and of the continuity assumptions made, that there exists a constant K' such that

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* R,Q,N,S all depend on p, but this dependence need not be explicitly indicated.

$$\left|W_p(z;z')\right| < K'/\left|X+iY\right| ;$$

hence, there exists a constant K such that

(3.13) $\left|W_p(z;z')\right| < K/\left|z-z'\right|$.

It is convenient here also to observe that, since equations (3.6) are satisfied by $S(z;z')$,

$$dS(z;z') = S_x(z;z')dx + S_y(z;z')dy = S_y(z;z')\ (dy - (A_p + e_{p,r_p-1})dx),$$

and, hence,

(3.14) $W_p(z;z')\ (dy - (A_p + e_{p,r_p-1})dx)$

$$= (2\pi i)^{-1}\ e^{it(Q(z;z') - Q(z';z'))}\ e^{t(N(z;z') - N(z';z'))}\frac{S_y(z';z')}{S_y(z;z')}\ \frac{dS(z;z')}{S(z;z')} .$$

Let us now wirte Green's formula

(3.15) $$\iint_{D_k - \omega_\gamma} H_p(V)\ W_p(z;z')\ dxdy = \left\{\int_{T_k} - \int_{\gamma}\right\} W_p V_p(dy - (A_p + e_{p,r_p-1})dx)$$

for the domain obtained from D_k by deleting the interior ω_γ of a small circle γ about the fixed point $z' \varepsilon D_k$. The limit of the area integral as γ shrinks to z' exists in view of (3.13). The limit of the integral over γ tends to $V_p(z')$, as may be shown from (3.14) by the procedures used in Theorem 1.1. Hence,

(3.16) $$V_p(z') = \int_{T_k} W_p V_p(dy - (A_p + e_{p,r_p-1})dx) - \iint_{D_k} H_p(V) W_p\ dxdy.$$

This expression corresponds entirely to equation (18) in Carleman's paper, and from now on the argument is exactly like Carleman's (pp. 6-9). The extension from totally elliptic systems to systems of mixed elliptic and hyperbolic type also can be carried out in the present case just asCarleman had propsed for his somewhat more special system.

CHAPTER III

BOUNDARY PROBLEMS

1. Extremum principle and uniqueness theorems. If two functions $u(y,y)$, $v(x,y)$, which are of class C' in a domain R, there satisfy the Cauchy-Riemann equations:

$$u_x - v_y = 0$$
$$u_y + v_x = 0,$$

neither function can be greater (less) than the maximum (minimum) of the function on the boundary. Proof is usually based upon the fact that u and v each satisfies a certain secon-order elliptic equation, namely, $u_{xx} + u_{yy} = 0$, whose solutions previously are known to possess the extremum property in question.

Functions $u(x,y)$, $v(x,y)$ of class C' in a domain R in which

(1.1) $$(D_x + (b' + ib'')D_y)(u + iv) = 0 \qquad (b'' \neq 0 \text{ in } R)$$

also are subject to this extemum principle, at least if b', b'' are sufficiently smooth functions of (x,y) in R. On the assumption, for instance, that $b = b' + ib''$ is Hölder-continuous in each closed subdomain of R, by a suitable change of variables $X = X(x,y)$, $Y = Y(x,y)$ the canonical elliptic system (1.1) can be reduced to the Cauchy-Riemann equations

$$(D_X + iD_Y)(u + iv) = 0;$$

thereby, extremum properties of solutions of a system of first-order equations again are derived from previously known extremum properties of solutions of second-order equations.

A different kind of argument can be given which utilizes no prior

knowledge about second-order equations, and, also not involving any change of variables, applies to systems of the form (1.1) in which b is merely continuous in R. The argument will be developed in a series of lemmas.

LEMMA 1.1 Let

$$u_1 = F_1(x_1,\dots,x_n),\dots, u_n = F_n(x_1,\dots, x_n)$$

map a bounded domain R of Euclidean n-space into this n-space. It is assumed that each F_i is of class C' in the closure of R and, in addition, that the functional determinant

$$J(x) = J(x_1,\dots,x_n) = \frac{\partial(u_1,\dots, u_n)}{\partial(x_1,\dots,x_n)}$$

is non-negative in R. Then the image F(S) of the set

$$S = \left[x \in R \;\middle|\; J(x) = 0\right]$$

is nowhere dense in the image F(R) of R.

Proof: It suffices to show that F(S) has exterior measure zero. Indeed, S is closed, hence F(S), its continuous image, also is closed, whence it follows that if F(S) were dense in some sphere K, F(S) would contain K and not, thus, be of measure zero.

Choose $\varepsilon > 0$. Let $N = N_\varepsilon$ be an open set covering F(S) such that, at any point x of the inverse image T of N, $|J(x)| < \varepsilon$.

The measure of the open set T is bounded uniformly with respect to ε, say

$$|T| \leqslant B,$$

where B is the measure of the bounded set R. The measure of N, the

image of T, is at most

$$\int_T J(x)\ dx \leqslant \mathcal{E} \int_T dx \leqslant \mathcal{E}\ B,$$

where dx represents the volume element in the n-dimensional x-space considered. It follows that the measure of $N_{\mathcal{E}}$ can be made arbitrarily small by sufficiently reducing $\mathcal{E}$, i.e., that F(S) has outer measure zero.

LEMMA 1.2. Under the hypotheses of Lemma 1.1, let U be an open, full sphere contained in the image of R: $U \subset F(R)$. Assume the closure of U is disjoint from the image of the boundary $\dot{R}$ of R: $\bar{U} \cap F(\dot{R}) = 0$. Then in U the degree of mapping of R into F(R) is positive.

<u>Proof:</u> Let X be any component in R of the set antecedent to U. Since U contains no image point of the boundary of R, $F(X) \subset U$, $F(\dot{X}) \subset \dot{U}$ * : i.e., F maps X into U, the boundary of X into the boundary of U. The degree of this mapping of X into U, called a local mapping is, therefore, constant in U. By Lemma 1.1, U contains a full sphere U_1 which does not intersect F(S): i.e., $F(x) \in U_1$ implies $J(x) > 0$. Let $X_1, X_2, \ldots$ be the components in X of the open set which is antecedent to U_1. Thus, $X_i \subset X$, $F(X_i) \subset U_1$, and $F(\dot{X}_i) \subset \dot{U}_1$. As we have seen, $J(x) > 0$ in each X_i; it follows that the local degree of mapping of X_i into U_1 is positive for every i, and the same is, therefore, true of the degree of mapping in U_1 of X into U. But the degree

* The boundary of any point set V is represented as $\dot{V}$.

of mapping of X into U is constant and, thus, also must be positive. We have thus shown that the local degree of mapping in U of any component of its inverse image is necessarily positive, and from this it follows immediately that the global degree of mapping in U of R into F(R) also is positive.

LEMMA 1.3. Let us make the hypotheses of Lemma 1.1 and assume, in addition, that for every point $\bar{x}$ of the boundary of R,

$$F_1(\bar{x}) < C,$$

where C is a constant. Then any open set contained in the range of F must lie in the half-space

$$u_1 \leqslant C.$$

Proof: A point $(\bar{u}_1,\ldots,u_n)$ for which $u_1 > C$ is in the outer component of the complement of the image of R; hence, the degree of mapping over a neighborhood of such a point is zero. By Lemma 1.2, this neighborhood was not covered by the mapping.

LEMMA 1.4. Let

$$u_i = F_i(x_1,\ldots, x_n) \qquad (i = 1,\ldots, n),$$

(or, for short, $u = F(x)$) where the F_i are of class C' in the closure of a bounded domain R. Suppose that the functional determinant

$$J(x) = \frac{\partial(u_1,\ldots, u_n)}{\partial(x_1,\ldots, x_n)}$$

is non-negative in R, and, moreover, that J(x) can vanish in R only if each partial derivative $\frac{\partial u_i}{\partial x_j}$ also vanishes. If, finally, each point $\bar{\bar{x}}$ of the boundary of R is mapped into the half-space $u_1 \leqslant c$ (c = const):

$$F_1(\bar{x}) \leqslant c \qquad (\bar{x} \in \dot{R}),$$

then each point $x \in R$ also is mapped into this half-space:

$$F_1(x) \leq c \qquad (x \in R).$$

Proof: Let T be any component of the inverse image of that part of $F(R)$ in which $u_1 > c$. Because the mapping is continuous, $u_1 = c$ on the image of the boundary of T. Also, $J(x) = 0$ in T, for otherwise $F(T)$ would contain an open set, a contradiction to Lemma 1.3. Hence, $u = F(x)$ is constant in T and is, thus, a point for which $u_1 = c$. But this contradicts the definition of T, unless T is empty.

THEOREM 1.1. Let R be a bounded domain in the xy-plane in which the functions $u(x,y)$, $v(x,y)$ are defined and of class C' and satisfy the elliptic system of partial differential equations

$$(1.2) \qquad (D_x + bD_y)(u + iv) = 0,$$

where $b(x,y) = b'(x,y) + ib''(x,y)$ is continuous, and $b''(x,y) > 0$, in R. Suppose, further, that $u(x,y)$ is continuous in the closure of R, and that, at all points $(\bar{x},\bar{y})$ of the boundary of R, $u(\bar{x},\bar{y}) \leq 0$. Then $u(x,y) \leq 0$ at all points $(x,y) \in R$.

Proof: From the given system of equations, which may be written

$$u_x + b'u_y - b''v_y = 0,$$
$$v_x + b'v_y + b''u_y = 0,$$

we see

$$J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x + b'u_y & u_y \\ v_x + b'v_y & v_y \end{vmatrix} = b''(u_y^2 + v_y^2).$$

Lemma 1.4 thus applies to any component T_c of the open set $\left[(x,y) \in R \;\middle|\; u(x,y) > c \right]$ $(c > 0)$, a set on whose boundary $u = c$. By this lemma, therefore, $u(x,y) \leq c$ in T_c, a contradiction if T_c is not

vacuous. Since c can be arbitrarily small, the conclusion of the theorem follows.

THEORDEM 1.2. Suppose that all the hypotheses of Theorem 1.1 hold and, in addition, that $u = 0$ on the boundary of R. Then $u(x,y)$ $v(x,y)$ are both identically constant in R.

<u>Proof:</u> By Theorem 1.1, $u(x,y) \leq 0$ in R and, by a dual argument, $u(x,y) \geq 0$ in R; hence, $u \equiv 0$ in R. The differential equations then assert $v_x = 0$, $v_y = 0$ in R.

More generally, we have the

COROLLARY 1.2.1. Let R be a bounded domain in the xy-plane in which the functions $u_p(x,y)$, $v_p(x,y)$ $(p = 1,\ldots,r)$ are defined and of class C' and satisfy the elliptic system of differential equations

$$(1.3) \qquad (D_x + bD_y + e_{r-1}D_y) \sum_{1}^{r} (u_p + iv_p) = 0,$$

where $b(x,y) = b'(x,y) + ib''(x,y)$ is continuous, and $b''(x,y) > 0$, in R. Suppose, further that $u_p(x,y)$ $(p = 1,\ldots,r)$ is continuous in the closure of R, and that $u_p = 0$ on the boundary of R. Then $u_p(x,y)$, $v_p(x,y)$ are all identically constant in R.

<u>Proof:</u> Equations (1.3) are equivalent to

$$(D_x + bD_y)(u_p + iv_p) + D_y(u_{p+1} + iv_{p+1}) = 0 \qquad (p = 1,\ldots,r-1),$$
$$(D_x + bD_y)(u_r + iv_r) = 0.$$

Thus, by the boundary conditions and Theorem 1.2, $u_r + iv_r =$ const. If $u_{p+1} + iv_{p+1} =$ const., then $(D_x + bD_y)(u_p + iv_p) = 0$, and, again by applying Theorem 1.2, $u_p + iv_p =$ const.

THEOREM 1.3. Let R be a bounded domain in the xy-plane in which the functions $u(x,y)$, $v(x,y)$ are defined and of class C' and satisfy the elliptic system of equations (1.2), where $b(x,y) = b'(x,y) + ib''(x,y)$ is continuous, and $b''(x,y) > 0$ in R. Then $u(x,y)$ cannot be greater (less) at an interior point of R than its maximum (minimum) on the boundary. A dual statement holds for $v(x,y)$.

Proof: Theorem 1.1 is applied to any component of the domain

$$\Big[(x,y) \,\varepsilon\, R \;\Big|\; u(x,y) > \max_{(\bar{x},\bar{y})\,\varepsilon\,\dot{R}} u(\bar{x},\bar{y}) + \varepsilon\Big] \quad (\varepsilon > 0).$$

2. A boundary problem for canonical elliptic systems. Let $a(x) = a'(x) + ia''(x)$, $b(x) = b'(x) + ib''(x)$ be complex-valued uniformly H-continuous functions in a convex domain R_1 of x^1x^2-space, and let

$$M(z;D_x) \equiv a(z)D_{x^1} + b(z)\,D_{x^2} + e_{r-1}\,D_{x^2}.$$

Let

$$C_p(x) = \sum_{q=1}^{r} e_q(c'_{pq} + ic''_{pq}),\; D_p(x) = \sum_{q=1}^{r} e_q(d'_{pq} + id''_{pq}),\; G(x) = \sum_{p=1}^{r} e_p(g'_p + ig''_p)$$

be hypercomplex-valued functions H-continuous in R_1. Let R be a bounded domain whose closure is contained in R_1 and whose boundary $\dot{R}$ consists of a finite number of simple closed curves each with continuous curvature. With the dependent variables represented as a hypercomplex function

$$U(x) = \sum_{p=1}^{r} e_p(u_p + iv_p),$$

our object will be to solve in R the system of $2r$ equations

(2.1) $L(x,D_x)U \equiv M(x,D_x)U + \sum_{p=1}^{r} (C_p(x)\, u_p + D_p(x)\, v_p) = G(x)$

prescribing $u_p(x) = f_p(z)$ $(p = 1,\dots,r)$ on $\dot{R}$, the $f_p(z)$ being H-continuous functions there.

It can be assumed that

(2.2) $a'(x)\, b''(x) - a''(x)\, b'(x) = 1.$

With

(2.3) $X^1 = X^1(z;x) = -b'(z)\, x^1 + a'(z)\, x^2,$

$X^2 = X^2(z;x) = -b''(z)\, x^1 + a''(z)\, x^2,$

we then have the important identity

(2.4) $M(z;D_x) = i\left[D_{X^1} + iD_{X^2} + e_{r-1}\,(-ia'(z)\, D_{X^1} - ia''(z)\, D_{X^2})\right]$

in which the principal part of (4.1) is put into the form discussed in Chapter II. Proceeding as in Chapter II, section 1, we now introduce the hypercomplex function

$$t(z;X) = X^1 + iX^2 + \sum_{p=1}^{r-1} e_p\,(ia'(z))^{r-p-1}\, ia(z)\, X^1,$$

which is a generating solution of the equation

$$M(z;D_x)\, t(z;X) = 0.$$

We introduce also the elementary solution for the latter equation

$V(X,Y) = V(z;X,Y) = (2\pi i)^{-1} D_{X^1} t(z;X)\, (1 - e_{r-1} a''(a))^{-1}\, (t(z,X)-t(z;Y))^{-1},$

whose properties are as described in Chapter 2, Theorems 1.1 to 1.5.

Let

$$H(z;x,y) = -V(z;\, X(z;x),\, X(z;y))\ .$$

By Theorem 1.1 of Chapter 2, in view of (2.4),

(2.5) $\quad M(z;D_x)\, H(z;\, x,y) = 0.$

Moreover,

$$H(z;x,y) = \emptyset(z)\, \left(t\,(z;X(z;x)) - t(z;X(z;y))\right)^{-1}$$

$$= \emptyset(z)\, \Big[-b(z)\,(x^1-y^1) + a(z)\,(x^2-y^2)$$

$$+ \sum_{p=1}^{r-1} e_p(ia'(z))^{r-p-1} ia(z)\,(-b'(z)\,(x^1-y^1) + a'(z)\,(x^2-y^2)\Big]^{-1},$$

where $\emptyset(z)$ satisfies a uniform Hölder condition in R_1. From this can be deduced the important estimates -

$$(2.6) \qquad \begin{aligned} &H(z;x,y) = O(|x-y|^{-1}) \\ &H(y;x,y) - H(x;x,y) = O(|x-y|^{h-1}) \\ &\left[D_{x^k}(H(y;x,y) - H(x_o;x,y))\right]_{x=x_o} = O(|x_o-y|^{h-2}) \end{aligned}$$

and, by (2.5),

(2.7) $\quad L(x;D_x)\, H(z;x,y) = O(|x-y|^{h-2}),$

where h $(0 < h < 1)$ is not greater than the Hölder exponent for $a(z)$, $b(z)$ in R_1.

We shall approach the boundary problem in a fashion suggested by certain methods of E. E. Levi [1,2] and G. Giraud [3] developed to treat elliptic equations of the second order. Of central imporatnce will be

LEMMA 2.1. If $P(x)$ is a hypercomplex function which is bounded and absolutely integrable on R and satisfies a Hölder condition at an interior point x_o of R, then the integral

$$V(x) = \int_{R_1} P(y)\ H(y;x,y)\ dy \qquad (dy = dy^1\ dy^2)$$

has H-continuous first derivatives at x_o, and

$$(2.8) \quad L(x,D_x)\ V(x) = -P(x) + \int_{R_1} P(y)\ L(x,D_x)\ H(y;x,y)\ dy$$

at $x = x_o$.

Proof: We may write *

$$V(x) = \int_{R_1} P(y)\ H(x_o;x,y)\ dy + \int_{R_1} P(y)\ (H(y;x,y) - H(x_o;x,y))\ dy = V_1(x)$$

$+ V_2(x)$

By Theorem 1.4 of Chapter 2,

$$(2.9) \quad M(x_o,\ D_x)\ V_1(x) = -P(x_o).$$

To differentiate $V_2(x)$, we simply form difference quotients at $x = x_o$ and pass to the limit. The result, in viewof the estimates (2.6), is

$$(2.10) \quad D_{x^k} V_2(x)\Big|_{x=x_o} = \int_{R_1} P(y) \Big[D_{x^k}(H(y;x,y) - H(x_o;x,y))\Big]_{x=x_o} dy.$$

Further,

$$M(x_o,D_x)\ (H(y;x,y) - H(x_o;x,y)) = M(x_o,D_x)\ H(y;x,y)$$

by (2.5). Hence, and because also of the estimate (2.7),

$$M(x_o,D_x)\ V_2(x)\Big|_{x=x_o} = \int_{R_1} P(y) \Big[M(x_o,D_x)\ H(y;x,y)\Big]_{x=x_o} dy,$$

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* as in a similar proof given by E. Hopf [1] , Lemma 1, p. 203

from which, and from (2.9), formula (2.8) is completely justified.

That the first derivatives of $V(x)$ are H-continuous at x_0 follows from the H-continuity at that point of the derivatives of $V_1(x)$ and of $V_2(x)$. For $V_1(x)$, this statement has been proved in Theorem 1.4, Chapter 2; for $V_2(x)$, it follows from Theorem 3, Giraud [1], p. 373, in view of (2.6), (2.10).

As additional preparation for setting up the stated boundary problem in terms of a system of integral equations, we now introduce the following functions * :

$$K(x,y;0) = L(x,D_x)\ H(y;x,y),$$

$$K(x,y;m) = \int_{R_1} K(x,z;0)\ K(z,y;m-1)\ dz.$$

LEMMA 2.2. Each $K(x,y;m)$ is continuous in x and in y $(x,y \in R_1)$, if $x \neq y$, and for $(m+1)h < 2$,

$$K(x,y;m) = O(\ |x-y|^{(m+1)\ h-2}\ ).$$

If m is sufficiently large, $K(x,y;m)$ is continuous with respect to all x,y in R_1.

This lemma is an immediate consequence of Theorem 3, Giraud [2], p. 150.

LEMMA 2.3. If y is a fixed point of R_1, then $K(x,y;m)$ $(m = 0, 1,\ldots)$ is H-continuous with respect to x at each point x_0 of R_1 distinct from y.

* in partial analogy with the procedure of Giraud [3], pp. 22-23.

Proof: This statement is true for $m = 0$. To prove it in general, let $D \subset R_1$ be a disk about y of radius less than $(1/3)\,|x_0-y|$, and write

$$K(x,y;m) = \left\{ \int_D + \int_{R_1-D} \right\} K(x,z;0)\, K(z,y;m-1)\, dz = I'(x) + I''(x).$$

We may restrict x by requiring

$$|x - x_0| < (1/3)\,|x_0 - y| \quad .$$

Then for z in D, under this restriction, $K(x,z;0)$ satisfies a Hölder condition at x_0, say $|K(x,z;0) - K(x_0,z;0)| < K_1\,|x - x_0|^{\alpha}$, and, therefore, so does $I'(x)$:

$$|I'(x) - I'(x_0)| \leqslant \int_D |K(x,z;0) - K(x_0,z;0)|\;|K(z,y;m-1)|\, dz \leqslant K_1 |x-x_0|^{\alpha} \int_D |K(z,y;m-1)|\, dz$$

For z in $R_1 - D$, $K(z,y;m-1)$ is continuous by Lemma 2.2; hence, Lemma 2.1 applies to $I''(x)$.

Finally, we define

$$H(x,y;0) = H(y;x,y)$$

$$H(x,y;m) = H(y;x,y) + \sum_{q=1}^{m} \int_{R_1} H(z;x,z)\, K(z,y;\, q-1)\, dz = H(y;x,y) + J(x,y;m).$$

Lemmas 2.3 and 2.1 show that

(2.11) $\quad L(x,D_x)\, H(x,y;m) = K(x,y;m).$

We also note

(2.12) $\quad J(xy;m) = O(\,|x-y|^{h-1}\,).$

Let us assume, in accordance with Lemma 2.2, that m is so large that the right side of (2.11) is continuous in (x,y) in R_1. We set

$$(2.13)\quad U(x) = -\int_{R_1} H(y;x,y)\,P(y)\,dy + \int_{\dot{R}} H(x,y;m)\,Q(y)\,(-b(y)\,dy^1 + a(y)dy^2 - e_{r-1}\,dy^1)$$

and attempt to determine hypercomplex functions

$$P(x) = \sum_{p=1}^{r} e_p(P'_p(x) + iP''_p(x)),$$

$$Q(z) = \sum_{p=1}^{r} e_p\,Q_p(z)$$

(P'_p, P''_p, Q_p real), defined and H-continuous for $x \in R_1$, $z \in \dot{R}$, respectively, such that U(x) is a solution of the stated problem. One set of 2r integral equations involving the 3r unknown real functions, obtained by applying Lemma 2.1, is

$$(2.14)\quad G(x) = P(x) - \int_{R_1} P(y)\,K(x,y;0)\,dy + \int_{\dot{R}} K(x,y;m)\,Q(y)\,(-b(y)\,dy^1 + a(y)\,dy^2 - e_{r-1}\,dy^1).$$

These equations are equivalent to the differential equations (2.1) assuming that the representation of the solution (2.13) is justified. They must be supplemented by r additional equations which expressthe boundary conditions

$$\sum_{p=1}^{r} e_p u_p(z) = \sum_{p=1}^{r} e_p f_p(z) \equiv f(z) \qquad (z \in \dot{R}),$$

equations which will follow from the behavior at the boundary of the second integral B(x) on the right side of (2.13). First, let us write

$$B(x) = \int_{\mathring{R}} H(y;x,y)\, Q(y)\, (-b(y)\, dy^1 + a(y)\, dy^2 - e_{r-1}\, dy^1)$$

$$+ \int_{\mathring{R}} J(x,y;m)\, Q(y)\, (-b(y)\, dy^1 + a(y)\, dy^2 - e_{r-1}\, dy^1) = B_1(x) + B_2(x)$$

and observe that $B_2(x)$ is continuous at $\mathring{R}$ because of (2.12). Secondly, if $x_o \in \mathring{R}$,

$$B_1(x) = \int_{\mathring{R}} H(x_o;x,y)\, Q(y)\, (-b(x_o)\, dy^1 + a(x_o)\, dy^2 - e_{r-1}\, dy^1)$$

$$+ \int_{\mathring{R}} H(x_o;x,y)\, Q(y)\, ((b(x_o) - b(y))dy^1 + (a(y) - a(x_o))\, dy^2$$

$$+ \int_{\mathring{R}} (H(y;x,y) - H(x_o;x,y))\, Q(y)\, (-b(y)\, dy^1 + a(y)\, dy^2 - e_{r-1}\, dy^1)$$

$$= B_{11}(x) + B_{12}(x) + B_{13}(x).$$

$B_{12}(x)$ is continuous at x_o in view of the H-continuity of $a(y)$, $b(y)$ at points of $\mathring{R}$.

$B_{13}(x)$, similarly, iscontinuous at x_o, because of the second estimate (2.6).

The third step is to change variables by

$$X = X(x_o;x), \qquad Y = X(x_o;y);$$

we then have

$$B_{11}(x) = F(X) = - \int_{\tilde{\mathring{R}}} \tilde{Q}(Y)\, V(X,Y)\, (dY^1 + i dY^2 + i e_{r-1}(-ia'(x_o)\, dY^2 + ia''(x_o)\, dY^1)).$$

where $Q(y) = \tilde{Q}(Y)$, and $\tilde{R}$ is the image of R. Since, by Theorem 1.5, Chapter II,

$$\lim_{X \to X_0} F(X) = F(X_0) + (1/2)\, \tilde{Q}(X_0),$$

where $X_0 = X(x_0; x_0)$, it follows that

$$\lim_{x \to x_0} B(x) = B(x_0) + (1/2)\, Q(x_0).$$

Thus, the desired supplmentary set of integral equations is

$$(2.15)\quad f(z) = (1/2)\, Q(z) - \int_{R_1} P(y)\; H(y;z,y)\, dy + \int_{\dot{R}} H(z,y;m)\, Q(y)\, (-b(y)\, dy^1 + a(y)\, dy^2 - \dot{e}_{r-1}\, dy^1).$$

We have shown that if a solution of our boundary problem exists and admits a representation of the form (2.13), where $P(x)$, $Q(z)$ are H-continuous in their respective domains, the latter functions must satisfy the system of integral equations (2.14), (2.15). If, conversely, these integral equations have a solution $P(x)$, $Q(z)$ $(x \in R_1,\ z \in \dot{R})$, then the latter functions are H-continuous *, and formula (2.13) defines a continuously differentiable function which solves the stated problem.

It remains to ascertain when the system of integral equations (2.14-15) has a unique solution. Clearcut general conditions are

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* This can be seen most easily after iteration. Also see Giraud [3] pp. 27-28.

apparently difficult to obtain **. And again following Giraud ***, I restrict attention to sufficiently small regions homothetic to a fixed region R_o whose boundary consists of a simple closed curve C having continuous curvature. We suppose the closure of R_o to be contained in a bounded convex domain R_1 in which, as required at the beginning of this section, the coefficients $a(x)$, $b(x)$, $C_p(x)$, $D_p(x)$, and the right hand side of equation (2.1) $G(x)$ are H-cotinuous. The prescribed boundary values $f(z)$ are required to be H-continuous on C. Assuming R_o to contain the origin let us change variables by

$$x^i = kt^i \qquad (k > 0),$$

$t^1 + it^2$ to range over R_o. Equations (2.1) then become

$$(2.1_k) \quad (a(kt)D_{t^1} + b(kt)D_{t^2} + e_{r-1}D_{t^2})U + k\sum_{p=1}^{r} (C_p(kt)u_p + D_p(kt)u_p) = kG(kt);$$

the corresponding integral equations will be designated (2.14_k) and (2.15_k). Reducing k is the same as homothetically shrinking R_o. If k tends to zero, equations (2.1_k) tend a system of entirely homogeneous equations (2.1_o) with constant coefficients. In case the Frecholm resolvent for the integral equations (2.14_o), (2.15_o) which correspond to (2.1_o) is regular (and leads in the limiting case, therefore, to a unique solutionfor the boundary problem), by continuity the resolvent for the integral equations

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** Attention is called, however, to the ingeious reasoning used by E. E. Levi [2] pp. 8-16, footnote p. 19, in his discussion of the Dirichlet problem. Such reasoning appears not to apply when a contour integral is involved.

*** [1], pp. 383-384.

(2.14_k-15_k) * also will be regular, if $k > 0$ is sufficiently small. Thus our problem is reduced to showing that a solution of equations (2.14_0) (2.15_0) is unique.

The coefficients in equations (2.1_0) being constant, we may assume these equations to be of the form

$$(D_{x^1} + iD_{x^2} + e_{r-1} \quad (aD_{x^1} + bD_{x^2}))U = 0.$$

The integral relations (2.14_0), (2.15_0) then reduce, respectively, to

$$P(x) = 0$$

and to

$$(2.16) \quad f(z) = (1/2)\, Q(z) - \int_C Q(y)\, V(z,y)(dy^1 + ie_{r-1}(ady^2 - bdy^1))$$

$$= (1/2)\, Q(z) - (2\pi i)^{-1} \int_C Q(y)\, d\log(t(y)-t(z))$$

$$= (1/2)Q(z) - (2\pi i)^{-1} \int_C Q(y) \quad d\log(y-z) + d \sum_{k=1}^{r-1} \frac{1}{k!} \left(\frac{T(y) - T(z)}{y-z} \right)^k \Big],$$

see Section 1, Chapter 2, What we have to show is that equation (2.16) has only the zero solution for $f(z) = 0$. Equating coefficients of e_r, we have

$$(2.17) \quad Q_r(z) = \pi^{-1} \int_C Q(y) \frac{d}{ds} R\left\{\frac{1}{i} \log (y-z)\right\} ds$$

$$= -\pi^{-1} \int_C Q(y) \frac{d}{dn} \log |y-z| \, ds,$$

---

* after sufficient iteration

whence * $Q_r(z) = 0$. Assuming $Q_r, \ldots, Q_{p+1} = 0$ $(p < r)$, we may next verify that Q_p satisfies an integral relation of the form (2.17) and, hence, also vanishes. It follows that $Q(z) = 0$ as was to be proved.

We remark that k is independent of the prescribed boundary values $f(z)$. k also is independent of $G(x)$, the right hand side of (2.1), a fact which follows from

LEMMA 2.4. Under the hypotheses stated at the beginning of this section, in any sufficiently small subdomain R_o of R_1, there exists a particular solution of class C' of equations (2.1).

Proof: Set

$$u(x) = \int_{R_o} P(y) \ H(y;x,y) \ dy.$$

The integral relations

$$G(x) = -P(x) + \int_{R_o} L(x,D_x)H(y;x,y) \ P(y) \ dy$$

then serve to determine $P(x)$ assuming R_o to be sufficiently small.

Let k be so small that, in accordance with the preceding, the integral equations (2.14_k), (2.15_k) have a unique solution $P(k;x)$, $Q(k;z)$. Estimates of these functions and of their Hölder coefficients can then be made ** from which by (2.13) we may obtain bounds for the solution $U(k;x)$ of the boundary problem for (2.1_k) and for its derivatives. Let G^o be an upper bound on R_1 for $G(x)$, G' the Hölder coefficient for $G(x)$ on R_1; let us assume $f(z) = 0$. Then there exist constants k_1, k_2, k_3, such that

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* See Courant-Hilbert, Vol. 2, pp. 269-270.

** The first of these estimates is furnished by Fredholm theory, the second by means such as are employed by Giraud [3] pp. 27-28.

$$(2.18) \qquad |U(k;x)| < k_1 G^0$$
$$|D_{x^s} U(k;x)| < K_2 G^0 + k_3 G',$$

where k_1, k_3 tend to zero with k.

This is an opportune place to insert proof of a statement required in section 3, Chapter 2, to the effect that, if $A(x,y)$ is a complex-valued non-real function of class C'' in a neighborhood of the origin in the xy-plane, then there is a disk K about the origin in which the equations

$$(2.19) \qquad (D_x + AD_y)\, w = 0$$

have a continuously differentiable solution $w(xy)$ whose first derivatives do not all vanish at the origin. * More specifically, we shall show there exist polynomials with complex coefficients

$$P(x,y) = p_1 x + p_2 y + p_{11}x^2 + p_{12}xy + p_{13}y^2 + p_{21}x^3 + p_{22}x^2 y + p_{23}xy^2 + p_{24}y^3,$$

$$Q(x,y) = q_{11}x^2 + 2q_{12}xy + q_{13}y^2 + q_{21}x^3 + q_{22}x^2 y + q_{23}xy^2 + q_{24}y^3,$$

p_1 being arbitrary, and there exists a continuously differentiable function $V(x,y)$, such that

$$U = P + QV$$

is a solution of (2.19).

To do so, we first write the expansion

$$A(x,y) = a_0 + a_1 x + a_2 y + a_{11}x^2 + a_{12}xy + a_{13}y^2 + R(x,y),$$

---

* We observe also that $w(x,y)$, like any other solution of (2.19) which is continuously differentiable in K, is of class C'' in the interior of K. This fact can be proved from Green's formula.

where a_0 is non-real, and

$R(x,y) = O((x^2 + y^2)^{3/2})$, $R_x(x,y) = O(x^2 + y^2)$, $R_y(x,y) = O(x^2 + y^2)$.

Given p_1, p_2 to satisfy $p_1 + a_0 p_2 = 0$, we evidently can determine the p_{jk} such that $(D_x + (A-R)D_y)P(x,y)$ is a sum of third order terms, i.e.,

$$(D_x + AD_y)P = R_1(x,y) = O((x^2 + y^2)^{3/2}).$$

Similarly, given q_{11}, q_{12}, q_{13} satisfying

$$q_{11} + a_0 q_{12} = 0, \quad q_{12} + a_0 q_{13} = 0.$$

the q_{2j} can then be so fixed as to assure

$$(D_x + AD_y)Q = R_2(x,y) = O((x^2 + y^2)^{3/2}).$$

Our actual choice of the q_{1k} will be such that

$$(2.20) \qquad \left|q_{11}x^2 + 2q_{12}xy + q_{13}y^2\right| \geqslant c(x^2 + y^2) \qquad (c > 0), *$$

a condition which guarantees the Lipschitz-continuity in K of the functions

$$b_1(x,y) = R_1(x,y)/Q(x,y), \quad b_2(x,y) = R_2(x,y)/Q(x,y),$$

assuming K to be sufficiently small.

To prove our statement, it is now obviously enough to produce a continuously differentiable solution of the linear equation

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* This is easily done. Assuming, without loss of generality, that $a_0 = i$, we may, for instance, take $q_{11} = 1$, $q_{12} = i$, $q_{13} = -1$. The quadratic form in question then becomes

$$x^2 - y^2 + 2ixy = (x + iy)^2.$$

$$(D_x + AD_y)V + b_2 V + b_1 = 0.$$

This can be done for sufficiently small K by means of a suitable integral representation as employed above.

3. <u>A problem for systems of differential equations of mixed elliptic and hyperbolic type.</u> For each $j = 1, \ldots, s$, let

(3.1) $$E^j(U^j)$$

be an elliptic expression in

$$U^j = \sum_{k=1}^{r_j} e_k^{(2r_j)} (u^{jk} + iv^{jk})$$

of the same type as the left side of (2.1). Its coefficients will be assumed to be uniformly H-continuous in a domain R_1 of the x^1x^2-plane. Let C be a simple closed curve having continuous curvature which with its interior R is contained in R_1 and is such that

(1) each system of equations

(3.2) $$E^j(U^j) = G^j(x) \qquad (j = 1, \ldots, s),$$

the $G^j(x)$ being uniformly H-cotinuous in R_1, has one and only one solution in R such that u^{jk} $(k = 1, \ldots, r_j)$ assumes arbitrarily prescribed H-continuous valueson C.

Let $K^j(x, w^1, \ldots, w^M)$, defined for $x \in R_1$ and for all $w^1, \ldots, w^M$, satisfy the inequality

$$\left| K^j(x_1, w_1^1, \ldots, w_1^M) - K^j(x_2, w_2^1, \ldots, w_2^M) \right| < K^0 \max_{p,\, q} |w_p^q| \, |x_1 - x_2|^\alpha + K' \sum_{p=1}^{M} |w_1^p - w_2^p|$$

$$(0 < \alpha \leqslant 1).$$

We shall suppose R to be such that

(2) the solution of (3.2) for which the prescribed boundary values vanish satisfies

$$(3.3)\qquad \left|U^j(x)\right| < k_1 G^0$$

$$\left|D_{x^s} U^j(x)\right| < k_2 G^0 + k_3 G',$$

G^0 being an upper bound on R_1 for $|G(x)|$, G' the Hölder coefficient for $G(x)$ on R_1, where

$$k_1, k_3 < C \ \text{Max}\left(1, \frac{1}{HK'}\right) \qquad (0 < C < 1).$$

Let

$$(3.4)\qquad H^j(w) = \sum_{k=1}^{t} p^{jk}(D_{x^1} + A^j D_{x^2} + B^j)\, w^k \qquad (j = 1,\dots,t)$$

be a hyperbolic system of t linear expressions in $w = (w^1,\dots,w^t)$. We shall suppose the coefficients of (3.4) to be of class C' in R_1. We suppose also that

(3) the hyperbolic system of equations

$$(3.5)\qquad H^j(w) = F^j(x),$$

F^j being of class C' in R_1, has one and only one continuously differentiable solution in R_1 which coincides on a fixed initial curve I with arbitrarily prescribed continuously differentiable initial functions.

Let $L^j(x,w^1,\dots,w^M)$, defined and of class C' for $x \in R_1$ and for all $w^1,\dots,w^M$, satisfy the inequality

$$\left|L^j(x,w_1^1,\dots,w_1^M) - L^j(x_2,w_2^1,\dots,w_2^M)\right| < L^0 \,\text{Max}\left|w_p^q\right| \left|x_1-x_2\right| + L' \sum_{p=1}^{M} \left|w_1^p - w_2^p\right| .$$

We shall suppose, finally, that

(4) the solution of (3.5) for which the prescribed initial values vanish satisfies

$$(3.6)\quad |w^j(x)| < c_1 \max_{x \in R_1} |f^q(x)|$$

$$\left| D_{x^s} w^j(x) \right| < c_2 \max_{x \in R_1} \left(|f^q(x)| , \ \left| D_{x^k} f^q(x) \right| \right),$$

where

$$c_1, c_2 < \text{Max}\left(c, \quad c/ML' \right) \qquad (0 < c < 1)$$

We shall show, under the foregoing hypotheses, that the equations

$$(3.7)\qquad E^j(U^j) = K^j(x, w^1, \ldots, w^M)$$

$$H^k(w) = L^j(x, w^1, \ldots, w^M),$$

where the symbols $w^{t+1}, \ldots, w^M$ stand for the u^{jk}, v^{jk}, have one and only one solution such that the u^{jk} $(k = 1, \ldots, r_j)$ assume arbitrarily prescribed H-continuous values on C and $w^1, \ldots, w^t$ arbitrarily prescribed continuously differentiable initial values on the initial curve I.

Let us write the system in more abbreviated form as

$$E(w) = K(x, w)$$

$$H(w) = L(x, w).$$

Let w_1, w_2, ... be a sequence of functions (having M components as above) which are continuous and have uniformly bounded first derivatives in R_1-C and which satisfy the stated initial and boundary conditions and the equations

$$E(w_{n+1}) = K(x,w_n(x))$$
$$H(w_{n+1}) = L(x,w_n(x)) \qquad (n = 1,2,\dots)$$

in R_1-C. With the notation

$$N_1(w) = \operatorname*{Max}_{\substack{p=1,\dots,t \\ x \in R_1-C}} |w^p(x)|, \quad N_2(w) = \operatorname*{Max}_{\substack{p=t+1,\dots,M \\ x \in R_1-C}} |w^p(x)|, \quad N(w) = \operatorname{Max}(N_1(w), N_2(w))$$

we have from (3.3), by considering the equation

$$E(w_{n+1} - w_n) = E(w_{n+1}) - E(w_n) = K(x,w_n(x)) - K(x,w_{n-1}(x)),$$

the estimate $N_2(w_{n+1}-w_n) < k_1 K' \sum_{p=1}^{M} | w^p_{n+1} - w^p_n | < cN(w_n - w_{n-1})$.

Similarly, $N_1(w_{n+1} - w_n) < cN(w_n - w_{n-1})$; hence

(3.8) $\qquad N(w_{n+1} - w_n) < cN(w_n - w_{n-1}) < c^n W_o,$

where W_o is a constant. Using the second estimate of (3.3), we have, further,

$$N_2(D_{x^s}(w_{n+1}-w_n)) < k_2 K' \sum_{p=1}^{M} |w^p_{n-1} - w^p_n| + k_3 \Big[K^o N(w_n - w_{n-1}) + K' \sum_{p=1}^{M} \operatorname*{Max}_{x \in R_1-C}$$
$$| D_{x^s} (w^p_{n-1} - w^p_n) | < k_4 N(w_n - w_{n-1}) + cN(D_{x^s}(w_n - w_{n-1})).$$

Further computation using the second estimate of (3.6) then shows, in fact, that

(3.9) $\qquad N(D_{x^s}(w_{n+1} - w_n)) < k_5 N(w_n - w_{n-1}) + cN(D_{x^s}(w_n - w_{n-1})).$

An immediate consequence of the inequalities (3.8), (3.9) is the uniform convergence in R_1-C of the sequence (w_n) and of the sequences of its partial derivatives. This completes the proof.

BIBLIOGRAPHY

A. A. Albert. [1] Modern Higher Algebra. University of Chicago Press, Chicago, 1937.

T. Carleman [1] , "Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes," Arkiv för Matematik, Astronomi och Fysik, Vol. 26B, No. 17, pp 1-9 (1939)

R. Courant [1] . Geometrische Funktionentheorie. Springer, Berlin, 1925.

R. Courant and D. Hilbert, [1] Methoden der Mathematischen Physik. Springer, Berlin, 1937. Distributed by Interscience Publishers, Inc., New York.

R. Courant and P. Lax, [1] "On nonlinear partial differential equations with two independent variables," Communications on Pure and Applied Mathematics 2:255-274 (1949)

J. B. Diaz, [1] "On a class of partial differential equations of even order," American Journal of Mathematics 68:611-659 (1946)

K. O. Friedrichs, [1] "Nonlinear hyperbolic differential equations for functions of two independent variables," American Journal of Mathematics 70:555-589 (1948)

G. Giraud, [1] "Sur les équations aux dérivées partielles du type elliptique," "Bulletin des Sciences Mathematiques (Series 2) 53:367-395 (1929).
[2] "Sur le problème de Dirichlet généralisé," Annales scientifiques de l'École Normale Supérieure (Series 3) 46:131-245 (1929).
[3] "Sur certains problèmes non linéaires de Neumann et sur certains problèmes non linéaires mixtes," Annales scientifique de l'École Normale Supérieure (Series 3) 49:1-104 (1932).

H. G. Häfeli, [1] "Hyperkomplexe Differentiale," Commentarii Mathematici Helvetici 20:382-420 (1947)

D. Hilbert, [1] "Grundzuege einer allgemeinen Theories der linearen Integralgleichungen," Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalishche Klasse, Sechste Mitteilung, 1910.

E. Hopf, [1] "Ueber den funktionalen, insbesondere den analytischen Charakter der Loesungen elliptischer Differentialgleichungen Zweiter Ordnung," Mathematische Zeitschrift 34:194-233 (1932)

W. A. Hurwitz. [1] Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung. Inaugural-Dissertation, Goettingen, 1910.

E. E. Levi, [1] "Sulle equazioni lineari totalmente ellittiche alle derivate parziali," Rendiconti del circolo Matematico di Palermo 24:275-317 (1907).
[2] "I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali," Memorie di Matematica e di Fiscia della Societa Italiana delle Sciense (Series 3) 16:3-112 (1910).

I. G. Petrovskii, [1] "On some problems of the theory of partial differential equations," Trans lation No. 12, American Mathematical Society, 1949.

J. A. Ward, [1] "A theory of analytic functions in linear associative algebras," Duke Mathematical Journal 7:233-248 (1940).